Generalized Weak Structures

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Abstract

In this work we define generalized weak structures which naturally generalize minimal structures, generalized topologies and weak structures recently proposed in the literature. Moreover, we study some properties of the interior, the closure and other concepts in this new context.

Mathematics Subject Classification: Primary 54A05, Secondary 54D10

Keywords: generalized weak structure, interior, closure

Introduction

The study of more general structures than that of topological space has taken several directions over the past fifteen years. In 1996, Maki ([3]) studied minimal structures, or shortly $m$-structures, on a set $X$, i.e., collections for subsets of $X$ containing the empty set and $X$, with no other restriction. Since 1997, Császár has studied topological notions in collections which are closed under unions ([1]). They constitute the well-known generalized topologies. As a natural generalization of the above-mentioned structures, in 2011 this author ([2]) also introduces the weak structures, which are collections of subsets of $X$ containing the empty set. In addition, he defines interior and closure within this new context and shows important properties of these operations.

In this paper we define generalized weak structures as an extension of Császár’s weak structures. For them, we introduce interior, closure and other related notions. We certainly show that many properties of these ”familiar” notions remain valid under our more general assumptions.
1 The \(g\)-interior and \(g\)-closure

In this section we generalize all the results presented in [2] and particularly well-known results in generalized topological spaces ([1]) and minimal structures ([3] and [4]).

**Definition 1** A generalized weak structure (GWS) on the nonempty set \(X\), is a nonempty class \(g\) of subsets of \(X\).

If \(g\) is a generalized weak structure on \(X\), then each element of \(g\) is said to be \(g\)-open and the complement (in \(X\)) of a \(g\)-open set is a \(g\)-closed set.

It is clear that each generalized topology, each minimal structure and each weak structure is a GWS.

**Definition 2** Let \(g\) be a GWS on \(X\) and \(A \subseteq X\). The \(g\)-interior and the \(g\)-closure of \(A\) are defined by

\[ i_g(A) = \bigcup\{ U : U \subseteq A, U \in g \} \]

and

\[ c_g(A) = \bigcap\{ F : A \subseteq F, F^c \in g \} \]

respectively.

We note that in certain cases of GWS’s, it could happen that \(c_g(\emptyset) \neq \emptyset\) and \(i_g(X) \neq X\).

In the following result we show the main properties of the \(g\)-interior and the \(g\)-closure. In particular, we prove that the operators \(i_g\) and \(c_g\) are idempotent and monotonic.

**Proposition 3** Let \(g\) be a GWS on \(X\) and \(A, B \subseteq X\). The following properties hold:

1. \(i_g(A) \subseteq A\).
2. If \(A \in g\), then \(i_g(A) = A\).
3. If \(A \subseteq B\), then \(i_g(A) \subseteq i_g(B)\).
4. \(i_g(i_g(A)) = i_g(A)\).
5. \(A \subseteq c_g(A)\).
6. If \(A^c \in g\), then \(c_g(A) = A\).
7. If \(A \subseteq B\), then \(c_g(A) \subseteq c_g(B)\).
8. \(c_g(c_g(A)) = c_g(A)\).
Proposition 6 \( \subseteq \) \( A \in x_i \) example, \( c \) follows.

Example 4 Consider \( g \) \( \{a, b, c, d\} \) with \( g = \{\{a\}, \{b\}, \{c\}, \{d\}\} \). For \( A = \{a, c\} \) we have \( i_g(A) = A = c_g(A) \), but the set \( A \) is neither \( g \)-open nor \( g \)-closed.

The well-known relationship between the interior and closure remains valid in our case. Also, the classical characterization of these concepts in terms of its elements can be extended to GWS's.

Proposition 5 Let \( g \) be a GWS on \( X, A \subseteq X \) and \( x \in X \). The following statements hold:

1. \( i_g(A)^c = c_g(A^c) \).

2. \( i_g(A^c) = c_g(A)^c \).

3. \( x \in i_g(A) \), iff, there exists \( U \in g \) such that \( x \in U \subseteq A \).

4. \( x \in c_g(A) \), iff, \( U \cap A \neq \emptyset \) for all \( U \in g \) with \( x \in U \).

Proof. 1. \( i_g(A)^c = \cap \{U^c : A^c \subseteq U^c, (U^c)^c \in g\} = c_g(A^c) \).

2. It is enough to replace \( A \) by \( A^c \) in item 1.

3. It follows easily from Definition 2.

4. Suppose that there exists \( U \in g \) which contains \( x \) and \( U \cap A = \emptyset \). Then \( A \subseteq U^c \) and thus \( x \notin c_g(A) \). Conversely, if \( x \notin c_g(A) \) then there exists a \( g \)-closed set \( F \) with \( A \subseteq F \) and \( x \notin F \). So, \( x \in F^c \in g \) and \( F^c \cap A = \emptyset \). ■

Given a GWS \( g \) on \( X \) it is possible to consider functions from \( P(X) \) to \( P(X) \) with the form \( \phi_1 \circ \ldots \circ \phi_k \), where \( \phi_k = i_g \) or \( \phi_k = c_g \) for all \( k \). For example, \( i_g i_g = i_g \) and \( c_g i_g c_g = c_g \) as we have shown.

Proposition 6 Let \( g \) be a GWS on \( X \) and \( A \subseteq X \). Then \( i_g c_g i_g c_g = i_g c_g \) and \( c_g i_g c_g i_g = c_g i_g \).
Proof. Let $A \in P(X)$. It is clear that $i_\emptyset c_\emptyset(A) \subseteq c_\emptyset(A)$ and since $i_\emptyset$ and $c_\emptyset$ are idempotent and monotonic we have $c_\emptyset i_\emptyset c_\emptyset(A) \subseteq c_\emptyset(A)$ and $i_\emptyset c_\emptyset i_\emptyset c_\emptyset(A) \subseteq i_\emptyset c_\emptyset(A)$. On the other hand, $i_\emptyset c_\emptyset(A) \subseteq c_\emptyset i_\emptyset c_\emptyset(A)$ which implies that $i_\emptyset c_\emptyset(A) \subseteq i_\emptyset c_\emptyset i_\emptyset c_\emptyset(A)$. The remaining part is analogous. ■

In a similar way to [2] we can obtain several structures. If $\mathfrak{g}$ is a GWS on $X$ and $A \subseteq X$ we can define: $A \in \alpha(\mathfrak{g})$ iff $A \subseteq i_\emptyset c_\emptyset(A)$, $A \in \sigma(\mathfrak{g})$ iff $A \subseteq c_\emptyset i_\emptyset(A)$, $A \in \pi(\mathfrak{g})$ iff $A \subseteq i_\emptyset c_\emptyset(A)$, $A \in \rho(\mathfrak{g})$ iff $A \subseteq c_\emptyset i_\emptyset(A) \cup i_\emptyset c_\emptyset(A)$ and $A \in \beta(\mathfrak{g})$ iff $A \subseteq c_\emptyset i_\emptyset c_\emptyset(A)$.

In the following, we shall write $i$ for $i_\emptyset$ and $c$ for $c_\emptyset$ for a given GWS $\mathfrak{g}$.

Proposition 7 Let $\mathfrak{g}$ be a GWS on $X$ and $A \subseteq X$. Then, $\mathfrak{g} \subseteq \alpha(\mathfrak{g}) \subseteq \sigma(\mathfrak{g}) \subseteq \rho(\mathfrak{g}) \subseteq \beta(\mathfrak{g}) \, y \, \alpha(\mathfrak{g}) \subseteq \pi(\mathfrak{g}) \subseteq \rho(\mathfrak{g})$.

Proof. If $A \in \mathfrak{g}$, then $A = i(A) \subseteq ci(A)$. Since $i$ is monotonic and idempotent we have that $A = i(A) \subseteq ici(A)$. Thus $A \in \alpha(\mathfrak{g})$.

If $A \in \alpha(\mathfrak{g})$, then $A \subseteq ici(A) \subseteq ci(A)$. Thus $A \in \sigma(\mathfrak{g})$.

If $A \in \sigma(\mathfrak{g})$, then $A \subseteq ci(A) \subseteq ci(A) \cup ic(A)$. Thus $A \in \rho(\mathfrak{g})$.

If $A \in \rho(\mathfrak{g})$, then either $A \in \sigma(\mathfrak{g}) \subseteq \beta(\mathfrak{g})$ or $A \in ic(A)$. So, in any case $c(A) \subseteq cic(A)$ implying $A \subseteq cic(A)$. Thus $A \in \beta(\mathfrak{g})$.

We know that $i(A) \subseteq A$ and since $i$ and $c$ are monotonic we obtain that $ici(A) \subseteq ic(A)$. So, if $A \in \alpha(\mathfrak{g})$ then $A \subseteq ici(A) \subseteq ic(A)$. Thus $A \in \pi(\mathfrak{g})$.

It is evident that $A \in \pi(\mathfrak{g})$ implies $A \subseteq ic(A) \subseteq ci(A) \cup ic(A)$. Thus $A \in \rho(\mathfrak{g})$. ■

2 Other elementary concepts

Definition 8 Let $\mathfrak{g}$ be a GWS on $X$, $A \subseteq X$. The $\mathfrak{g}$—derived set of $A$, $d_\emptyset(A)$, is the set of all points $x \in X$ such that every $U \in \mathfrak{g}$ containing $x$ satisfies $(U \setminus \{x\}) \cap A \neq \emptyset$.

In the following proposition we prove that each $\mathfrak{g}$—closed set contains its $\mathfrak{g}$—derived set. Moreover, we use this last concept to characterize the $\mathfrak{g}$—closure.

Proposition 9 Let $\mathfrak{g}$ be a GWS on $X$ and $A, B \subseteq X$. The following statements hold:

1. If $A \subseteq B$, then $d_\emptyset(A) \subseteq d_\emptyset(B)$.

2. $d_\emptyset(A) \subseteq c_\emptyset(A)$.

3. If $A^c \in \mathfrak{g}$, then $d_\emptyset(A) \subseteq A$.

4. $c_\emptyset(A) = A \cup d_\emptyset(A)$.
**Proposition 12** Let $A \subseteq X$. The $g$-exterior and $g$-boundary of $A$ are defined as $e_g(A) = i_g(A^c)$ and $b_g(A) = c_g(A) \cap c_g(A^c)$ respectively.

**Example 10** Consider $\mathbb{R}$ with $g = \{[a, b] : a, b \in \mathbb{R}\}$. If $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, then $d_g(A) = \{0\} \subseteq A$. However, $A$ is not a $g$-closed set because $A^c \notin g$.

Similar to topological spaces we can define other concepts such as the boundary and exterior of a set.

**Definition 11** Let $g$ be a GWS on $X$ and $A \subseteq X$. The $g$-exterior and $g$-boundary of $A$ are defined as $e_g(A) = i_g(A^c)$ and $b_g(A) = c_g(A) \cap c_g(A^c)$ respectively.

**Proposition 12** Let $g$ be a GWS on $X$ and $A, B \subseteq X$. The following affirmations hold:

1. $e_g(A) \subseteq A^c$.
2. If $A^c \in g$, then $e_g(A) = A^c$.
3. If $A \subseteq B$, then $e_g(B) \subseteq e_g(A)$.
4. $b_g(A^c) = b_g(A)$.
5. If $A \in g$, then $A \cap b_g(A) = \emptyset$.
6. If $A^c \in g$, then $b_g(A) \subseteq A$.

We finish this section by presenting other properties that can be easily extended to generalized weak structures.

**Proposition 13** Let $g$ be a GWS on $X$ and $A \subseteq X$. The following statements hold:

1. $c_g(A) \setminus i_g(A) = b_g(A)$.
2. $X \setminus b_g(A) = i_g(A) \cup e_g(A)$.
3. $A \setminus b_g(A) = i_g(A)$.
4. $b_g(A) \cup A = c_g(A)$.
5. $b_g(A) \cup i_g(A) = c_g(A)$.
6. $X = i_g(A) \sqcup b_g(A) \sqcup e_g(A)$, where "\sqcup" denotes disjoint union.
3 $g$–subspaces

Definition 14 Let $g$ be a GWS on $X$ and $A \subseteq X$. The GWS on $A$ associated to $g$ is defined by the collection $g_A = \{G \cap A : G \in g\}$. In this case $A$ is called a $g$–subspace of $X$.

Note that if $A$ is a $g$–subspace of $X$ then $K \subseteq A$ is $g_A$–closed, if and only if, $K = F \cap A$ for some $g$–closed set $F$.

Proposition 15 Let $A$ be a $g$–subspace of $X$ and $B \subseteq A$. The following statements hold:

1. $d_{g_A}(B) = d_g(B) \cap A$.
2. $c_{g_A}(B) = c_g(B) \cap A$.
3. $i_{g_A}(B) \supseteq i_g(B) \cap A$.
4. $b_{g_A}(B) \subseteq b_g(B) \cap A$.

Proof. 1. Let $x \in d_{g_A}(B)$ and $H \in g$ such that $x \in H$. Then $H \cap A \in g_A$ and thus $((H \cap A) \setminus \{x\}) \cap B \neq \emptyset$ which implies that $(H \setminus \{x\}) \cap B \neq \emptyset$. Therefore, $x \in d_g(B)$ and we have the inclusion $d_{g_A}(B) \subseteq d_g(B) \cap A$. Let $x \in d_g(B) \cap A$ and $G \in g_A$ such that $x \in G$. Since $G = H \cap A$ with $H \in g$ we have that $(G \setminus \{x\}) \cap B = (H \setminus \{x\}) \cap B \neq \emptyset$. Thus $x \in d_{g_A}(B)$ and the other inclusion holds.

2. $c_{g_A}(B) = B \cup d_{g_A}(B) = B \cup (d_g(B) \cap A) = (B \cup d_g(B)) \cap (B \cup A) = c_g(B) \cap A$.

3. If $x \in i_g(B) \cap A$, then $x \in A$ and there exists $G \in g$ such that $x \in G \subseteq B$. So, $x \in A \cap G \subseteq B$ and consequently $x \in i_{g_A}(B)$.

4. It is consequence of item 2.

If $g$ is a GWS on $X$ and $B \subseteq A \subseteq X$, the set $B$ can be regarded as a $g$–subspace of $X$ with the GWS $g_B$ on $B$, or as a $g_A$–subspace of $A$ with the GWS $(g_A)_B$ on $B$.

Proposition 16 Let $g$ be a GWS on $X$ and $B \subseteq A \subseteq X$. Then $g_B = (g_A)_B$.

Proof. If $G \in g_B$, then there exists $H \in g$ such that $G = H \cap B$. So, $G = (H \cap A) \cap B \in (m_A)_B$ and hence $g_B \subseteq (g_A)_B$. The other inclusion is analogous.

Given a GWS $g$ on $X$ the property $\mathcal{P}$ on $X$ is called hereditary if every $g$–subspace of $X$ satisfies $\mathcal{P}$.

Example 17 Suppose that $g$ is a GWS on $X$ and $A \subseteq X$. If $g$ is closed under unions, then so is $g_A$. In fact, if $\{H_i \cap A\}_{i \in I}$ is a collection of elements of $g_A$, then $\bigcup_{i \in I}(H_i \cap A) = (\bigcup_{i \in I}H_i) \cap A \in g_A$ since $\bigcup_{i \in I}H_i \in g$. 
References


Received: May, 2012