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# A Note on the Spread of Infectious Diseases in a Large Susceptible Population

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**Abstract**

In this paper, we show that the recovery compartment of an epidemiological model for a disease in large susceptible population in an SIR model, is determined by the size of the population. We show that, for a large susceptible population, the transient distribution for the recovery compartment does not follow the usual a hyperbolic curve for a small population, but rather an exponential function. In addition, we show that the function for the recovered population has two equilibrium points which have characteristics of transcritical bifurcation.

**Mathematics Subject Classification:** 92D25; 92D30

**Keywords:** Analytic solution, exponential growth, recovered population, bifurcation, SIR model

# 1 Introduction

The susceptible-infectious-recovered (SIR) model has been used to describe the dynamics of the epidemics of several infectious diseases. A classical work which has drawn the attention of many epidemiologists over the years is Kermack-McKendrick [5] model which observed that a decrease in the susceptible individuals results in a decline in the effective reproductive number of the disease. Several modifications have been made this model. Of particular note is the work of Yorke and London [8] who included latent period of the infectious agent for ascertainment of susceptible, exposed, infectious and recovered (SEIR) model with seasonal force to describe irregular and biennial oscillatory of measles incidence. Anderson and May [1] observed that some diseases have temporary immunity so that recoveries gain their susceptibility when temporary immunity ended. Capasso and Serio [2] extended SIR model to include emigration-susceptibles. Just over a decade ago, Hethcote [4] used computer simulation to describe the derivative passive immune susceptibles, exposed individuals, infectious and recovered (MSEIR) model.

Naresh et al. [7] applied qualitative and numerical simulation for the dynamics of susceptible, infected and vaccinated with two different infectivities to show that the spread of an infectious disease increases as the carrier population density increases.

## 2 The SIR Model

Several methods of analysis have been used to extract the necessary information from the SIR system of equation and its variants that represent the spread of various diseases. Raggett [3] model of the epidemics of Eyam data (United Kingdom) shows that the recovered population grows exponentially. This is supported by the work of other researchers. An important observation from the analysis of the model in [5] is that the final sizes in the two population compartments are determined by the ratio of the dispersion rates of the individuals between the two compartments, see for example [6]. In this paper, we derive the function that generates the exponential growth curves of the recovered compartment for a large susceptible population, and derive an expression for recovered population. We further discuss equilibrium solution resulting from the approximate solution.

We review the SIR model for a large susciptible population. The SIR model is usually represented by the following system of differential equations.

$$\begin{aligned} \frac{dS}{dt} &= -\lambda SI \\ \frac{dI}{dt} &= \lambda SI - \sigma I \\ \frac{dR}{dt} &= \sigma I \end{aligned} \tag{1}$$

where the constant $\lambda > 0$ is the rate of infection and $\sigma > 0$ is the removal rate. This model is based on the assumption that the size of the population is constant, that there are no birth of deaths, immigration or emigration, that the infection rate $\lambda$ is proportional to the number of infectives and that the members of the population mix in a homogeneous manner. In addition, the model assumes that there is no latent period. Consequent to the above assumptions, the additional equation $S + I + R = N$, holds. Here, $N$ is the total population. This is verified by adding the three equations in the system (1) above to yield: $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.

To determine whether the infection will propagate in the affected population or not require either $\frac{dI}{dt} > 0$ or $\frac{dI}{dt} \leq 0$. The critical parameter that determines the sign of $\frac{dI}{dt}$, $R_0$, the basic reproductive ratio is given by $R_0 = \frac{\lambda S_0}{\sigma}$

The nondimentionalized system has one disease-free equilibrium at $(1, 0, 0)$. At the disease-free equilibrium, the following theorem holds:

**Theorem 2.1** *The disease-free equilibrium is globally asymptotically stable when $R_0 < 1$ and unstable if $R_0 > 1$. In addition,*
*(i) If $R_0 < 1$, the infected population dies out.*
*(ii) If $R_0 > 1$, the infected population persists.*

**Proof**: See for example [9].

In order to solve the nonlinear system (1), we divide the second equation by the first to give:

$$\frac{dI}{ds} = \frac{\sigma - \lambda S}{\lambda S} = \frac{\sigma}{\lambda S} - 1 \tag{2}$$

This differential equation is separable, and leads to the conservation equation:

$$S + I - \frac{\sigma}{\lambda} lnS = S_0 + I_0 - \frac{\sigma}{\lambda} lnS_0 \tag{3}$$

This can be rearranged as

$$I = -S + \frac{1}{R_0} lnS + C, \tag{4}$$

where $C$ is a constant. Taking limits as $t \to \infty$ we have $C = I_\infty + S_\infty - \frac{1}{R_0} lnS_\infty$. Hence $S_0 - S_\infty + \frac{1}{R_0} ln\frac{S_\infty}{S_0} = 0$. Setting $S_0 - S_\infty = x$, we have:

$$x + \frac{1}{R_0} ln(\frac{s_0 - x}{S_0} = 0 \tag{5}$$

Using the assumption that $S_0 = 1$, initially, we obtain:

$$x + \frac{1}{R_0} ln(1 - x) = 0 \tag{6}$$

Hence:

$$R_0 + \frac{1}{x} ln(1 - x) = 0 \tag{7}$$

This gives the equation for the final population size. It must be noted that if $R_0 > 1$ in the last equation, then $0 < x < 1$.

By the same technique, we have

$$\frac{dS}{dR} = -\frac{\lambda}{\sigma}S \Rightarrow S = S_0 e^{\frac{\lambda}{\sigma}(R-R_0)} > 0 \tag{8}$$

Solutions for $I$ and $R$ can now be obtained from the previous solutions. We note that since $S > 0, \forall t$, it follows that $S$ is never zero. This means that as long as individuals recover from the desease, it is impossible for the total population to be infected.

The recovered population is

$$\frac{dR}{dt} = \gamma(N - R - S_o \exp(-R/\rho)) \tag{9}$$

Approximating $\exp(-R/\rho)$ in equation (9) by it's second degree Taylor polynomial, the following solution is obtained:

$$\begin{aligned} \frac{dR}{dt} &= \gamma(N - S_o) + \gamma(\frac{S_o}{\rho} - 1)R - \frac{\gamma S_o}{2\rho^2}R^2 \\ R(t) &= \frac{\rho^2}{S_o}[\frac{S_o}{\rho} - 1 + \alpha \tanh(\frac{1}{2}\alpha\gamma t - \theta)] \end{aligned}$$

where $\alpha = [(S_o/\rho - 1)^2 + 2S_o(N - S_o)/\rho^2]^{1/2}$, $\theta = \tanh^{-1} 1/2(S_O/\rho - 1)$, and $\rho = \frac{\sigma}{\lambda}$. We note that the graph for the number of recoveries $R(t)$ is hyperbolic tangent. See figure 1.

At equilibrium $dR/dt = 0$, we obtain

$$\begin{aligned} R_1 &= (\rho - \frac{\rho^2}{S_o}) - \frac{\rho\sqrt{\rho^2 + 2S_oN - S_o^2 - 2S_o\rho}}{S_o} \\ R_2 &= (\rho - \frac{\rho^2}{S_o}) + \frac{\rho\sqrt{\rho^2 + 2S_oN - S_o^2 - 2S_o\rho}}{S_o} \end{aligned}$$

The roots of $f(R)$ are real and distinct since $\rho^2 + 2S_oN > S_o^2 + 2S_o\rho$ with expression under radical sign being the dominant term of equations (8)-(9). The physical quantity is greater or equal to zero hence $R_1$ should be $R_1 \geq 0$ and $R_2$ is positive for outbreak ($S_o > \rho$) or no outbreak ($S_o < \rho$) of the disease.

# 3 Large Population Approximation

For certain diseases in a large population the disease is introduced into the population when every member is practically disease-free. A typical example

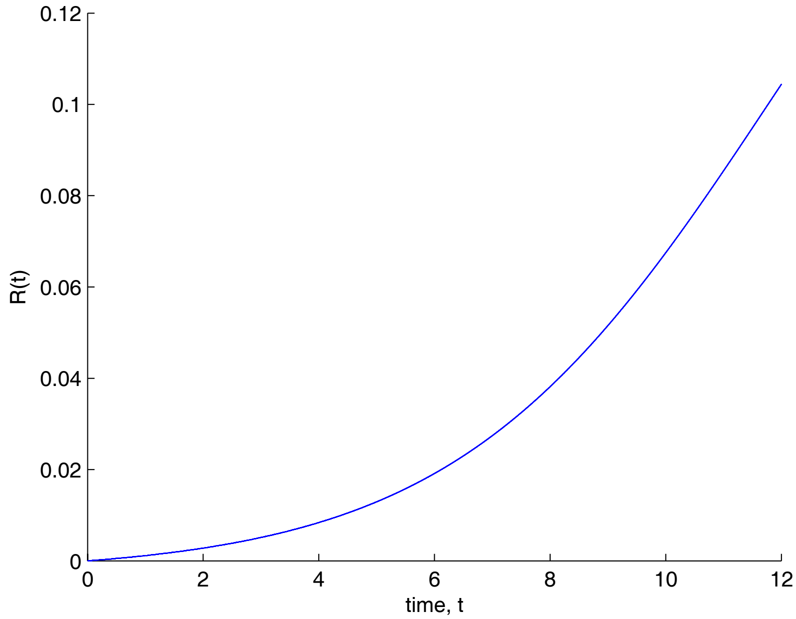


Figure 1: The number of recoveries for $R(t)$ for $R_0 > 1,\ \gamma = 0.05,\ \beta = 0.42$ and $S_o = 0.98$

is the spread of then unknown SARS in 2003 in several countries around the globe, or the diseases that are reintroduced into a country after their elimination by for example, vaccination. In such cases, the assumption $S_o \approx N$ at time $t = 0$ holds. By this assumption, equation (9) becomes

$$\begin{aligned} \frac{dR}{dt} &= -\gamma R + \gamma S_o(1 - \exp -(R/\rho)) \\ \frac{dR}{dt} &= \gamma(\frac{S_o}{\rho} - (\frac{S_o R}{2\rho^2} + 1))R = f(R) \end{aligned}$$

which is new equation for the rate of recovered population. Hence

$$\int \frac{2\rho^2}{R[2\gamma(\rho S_o - \rho^2) - \gamma S_o R]} dR = \int dt$$

Letting $a = 2\gamma(\rho S_o - \rho^2),\ \ b = \gamma S_o$ and $c = 2\rho^2$ yields

$$\begin{aligned} \int \frac{c}{R(a - bR)} dR &= \int dt \\ \frac{c}{a} \ln aR - \frac{c}{a} \ln(a^2 - abR) &= t + D \end{aligned}$$

$$R(t) = \frac{aA\exp(c/a)t}{1 + bA\exp(c/a)t}, \quad (10)$$

for all $bA \neq -1$. Replacing $a, b$, and $c$ in equation (10), and setting $A = \frac{a}{c}D$, we obtain

$$R(t) = \frac{2\gamma(\rho S_o - \rho^2)R_o \exp(\rho/(\gamma(S_o - \rho)))t}{1 + \gamma S_o R_o \exp(\rho/(\gamma(S_o - \rho)))t}. \quad (11)$$

provided $\gamma S_o R_o \neq -1$ and $R_o = A \neq 0$. $R(t) \to 2(\rho S_o - \rho^2)/S_o$ as $t \to \infty$. Figure 2 shows the plot of $R(t)$ for $R_o > 1$.

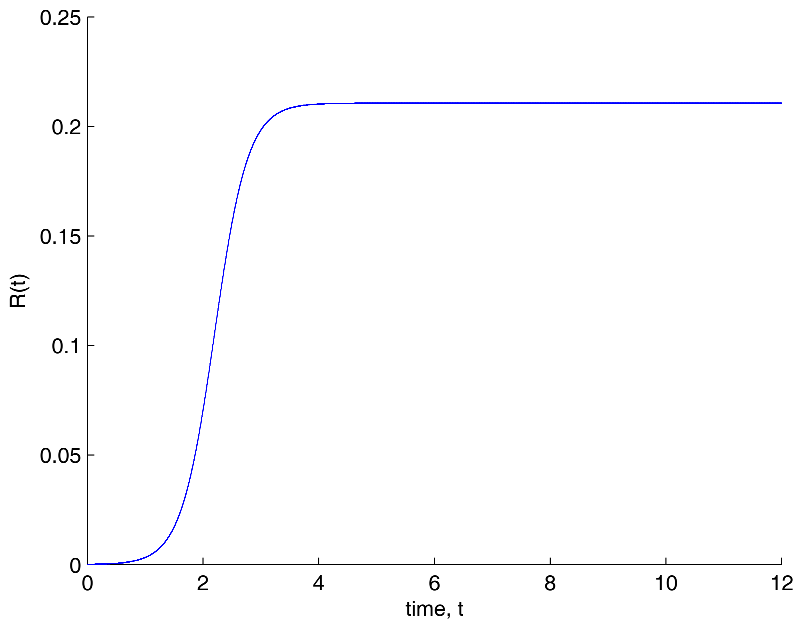


Figure 2: The number of infectives who recover as a functionof time

**Theorem 3.1** *Let $S_o > \rho$ and assume that the population size $N$ is large as compared to the initial number of susceptibles. Assume moreover, that difference between $N$ and $S_o$ is small, then the number of individuals who ultimately recovered/removed from the disease cannot exceed $\frac{\gamma}{2S_o}(S_o - \rho)^2$.*

**Proof**: We find the maximum number of susceptibles who recover from the disease.

$$f(r) = \frac{-1\gamma S_o}{2\rho^2}[r^2 - \frac{2\rho(S_o - \rho)r}{S_o} - \frac{2\rho^2(N - S_o)}{S_o} \quad (12)$$

$$\begin{aligned}
&= \frac{-1\gamma S_o}{2\rho^2}[(r - \frac{\rho(S_o-\rho)}{S_o})^2 - \frac{\rho^2(S_o-\rho)^2 - 2\rho^2 S_o(N-S_o)}{S_o^2} \\
&= \frac{-1\gamma S_o}{2\rho^2}(r - \frac{\rho(S_o-\rho)}{S_o})^2 + \gamma/2[\frac{(S_o-\rho)^2 + 2S_o(N-S_o)}{S_o}] \\
&= \frac{-1\gamma S_o}{2\rho^2}[r - (\rho - \frac{\rho^2}{S_o})]^2 + \frac{1\gamma}{2S_o}(\rho^2 + 2S_oN - S_o^2 - 2S_o\rho)
\end{aligned}$$

where

$$f_{max}(r) = \frac{\gamma}{2S_o}(\rho^2 + 2S_oN - S_o^2 - 2S_o\rho) \tag{13}$$

Since $N \approx S_o$ the maximum number of infected patients who have been recovered from their illness becomes

$$f_{max}(r) = \frac{\gamma}{2S_o}(S_o - \rho)^2 \tag{14}$$

Our analysis yields a single peak epidemic curve. We can easily see from above that an epidemic results when $(\rho - \frac{\rho^2}{S_o})$ is positive since $S_o > \rho$. If $R_o < 1$, it implies that $S_o < \rho$ then $(\rho - \frac{\rho^2}{S_o})$ is negative and stretches the epidemic curve parallel to negative $f(r)$ axis. $f_{max}(r)$ cannot be negative.

We now prove that $S_o^2 + \rho^2 > 2S_o\rho$ by considering the expression in parenthesis in equation (13). $N > S_o$ since $N = S + I + R$. We assume that $N \approx S_o$ then

$$\rho^2 + 2S_oN - S_o^2 - 2S_o\rho$$

becomes

$$\begin{aligned}
&\rho^2 + 2S_o^2 - S_o^2 - 2S_o\rho \\
&\Rightarrow \rho^2 + S_o^2 - 2S_o\rho \\
&\Rightarrow (S_o - \rho)^2 \geq 0.
\end{aligned}$$

Computing the norm of the above expression, we have:

$$\begin{aligned}
\|S_o - \rho\|^2 &\geq 0 \\
\|S_o - \rho\|^2 &= \langle S_o - \rho, S_o - \rho\rangle \\
&= \langle S_o, S_0\rangle - \langle S_o, \rho\rangle - \langle \rho, S_o\rangle + \langle \rho, \rho\rangle \\
&= \langle S_o, S_o\rangle - \langle S_o, \rho\rangle - \overline{\langle S_o, \rho\rangle} + \langle \rho, \rho\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \quad \|S_o\|^2 - 2Re\langle S_o, \rho\rangle + \|\rho\|^2 \\
&= \quad \|S_o\|^2 - 2|\langle S_o, \rho\rangle| + \|\rho\|^2 \\
&\leq \quad \|S_o\|^2 - 2\|S_o\|\|\rho\| + \|\rho\|^2 \\
&\Rightarrow \quad \|S_o\|^2 - 2\|S_o\|\|\rho\| + \|\rho\|^2 \geq 0 \\
&\Rightarrow \quad \|S_o\|^2 + \|\rho\|^2 \geq 2\|S_o\|\|\rho\| \\
\|S_o\| \neq \|\rho\| & \\
&\Rightarrow \quad \|S_o\|^2 + \|\rho\|^2 > 2\|S_o\|\|\rho\| \\
&\Rightarrow \quad S_o^2 + \rho^2 > 2S_o\rho
\end{aligned}
$$

Also,

$$\rho^2 + 2S_oN > S_o^2 + 2S_o\rho$$

Since the expression on the left hand side is greater than that on right hand side the maximum number that recovers is always positive, as expected. The term

For $f(r) = \frac{\gamma}{2S_o}(S_o - \rho)^2$, the equation $f(r) = 0$ has a single root $r = (\rho - \frac{\rho^2}{S_o})$. We now determine the convergence of the series solution.

$$
\begin{aligned}
\lim_{n\to\infty} \|\frac{r_{n+1}}{r_n}\| &= \lim_{n\to\infty} \|\frac{(-1)^{n+2}\gamma S_o r^{n+1}}{(n+1)!\rho^{n+1}} \times \frac{n!\rho^2}{(-1)^{n+1}\gamma S_o r^n}\| \\
&= \lim_{n\to\infty} \|\frac{(-1)r}{\rho(n+1)}\| \\
&= \|\frac{r}{\rho}\| \lim_{n\to\infty} \frac{1/n}{1+1/n} \\
&= \frac{r}{\rho} \times 0 = 0 < 1
\end{aligned}
$$

# 4 Results

We note that the rate function is given by $f(R) = \gamma(\frac{S_o}{\rho} - (\frac{S_oR}{2\rho^2} + 1))R$ and $f(R) = 0$ gives equilibrium points to

$$
\begin{aligned}
R_1^* &= 0 \\
R_2^* &= \frac{2(S_o\rho - \rho^2)}{S_o}
\end{aligned}
$$

Table 1: sign of the slope for $f(R) = \gamma(\frac{S_o}{\rho} - (\frac{S_o R}{2\rho^2} + 1))R$ for the intervals $(-\infty, R_1^*)$, $(R_1^*,\ R_2^*)$, $(R_2^*, \infty)$

| Interval | Sign of f(R) | R(t) |
|---|---|---|
| $(-\infty, R_1^*)$ | negative | decreasing |
| $(R_1^*, R_2^*)$ | positive | increasing |
| $(R_2^*, +\infty)$ | negative | decreasing |

### 4.1 Bifurcation of the number of Recoveries from the disease

When $S_o < \rho$, the equilibrium point $R = R_2^*$ becomes ustable and $R = R_1^*$ is stable. As $S_o$ increases, the $R_2^*$ approaches the origin, and coalesces with $R_1^*$ when $S_o = \rho$. Finally, $R_2^*$ becomes stable equilibrium point and $R_1^*$ becomes unstable equilibrium point. Thus the equilibrium points exchange stabilities as $S_o$ is varied. The system undergoes transcritical bifurcation at equilibrium $S_o = \rho$. See figure 3.

## 5 Conclusion

It can be observed that from our results that for la large susceptible population, the recovered class follows an exponential function as shown in fig 1 and as shown by the numerical simulations of [3] and others. We showed that for a large susceptible population, the transient distribution for the recovery compartment does not follow the usual a hyperbolic curve for a small population, but rather an exponential function. In addition, we show that the function for the recovered population has two equilibrium points with characteristics of transcritical bifurcation.

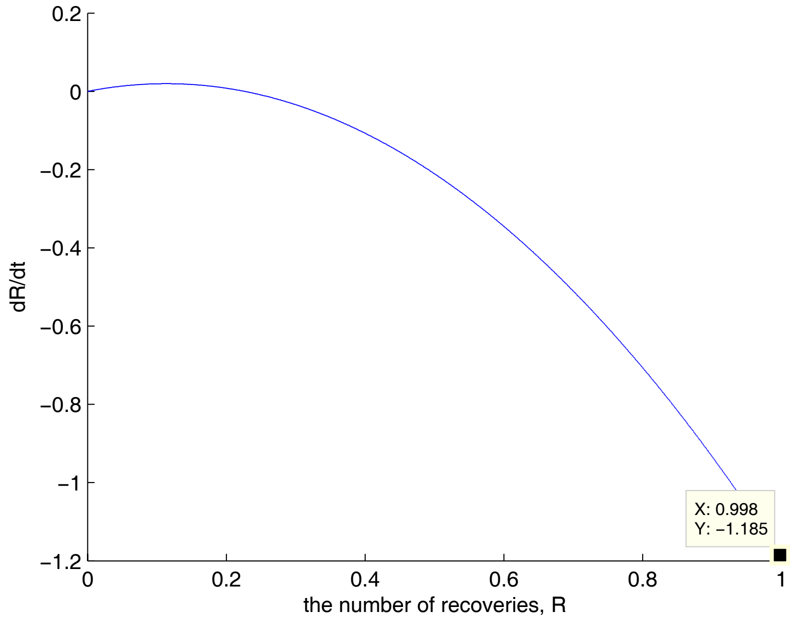


Figure 3: The transcritical bifurcation for $S_0 = 1, \rho = 1.3 \times 10^{-1}$

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