International Mathematical Forum, Vol. 7, 2012, no. 40, 1975 - 1979

Group Inverse for a Class 2×2 Circulant Block Matrices over Skew Fields

Junqing Wang

Department of Mathematics, Dezhou University
Shandong 253055, China
tjgdwjq@sina.com

Abstract. In this paper, we give the existence and the representation of the group inverse for circulant block matrix $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} (A, B \in K^{n\times n}, and A^2 = A, B^2 = B)$ over skew field . Some relative additive results are also given.

Mathematics Subject Classification: 47H09; 47H10

Keywords: skew; circulant block matrix ;group inverse

1. Introduction

Let K be a skew field and I be the Unit matrix . $K^{n\times n}$ and A^* respectively denote the set of all n-order matrices and the conjugate transpose of A over K. For$A \in K^{n\times n}$, the matrix $X \in K^{n\times n}$is said to be the group inverse of A, if $AXA = A, XAX = X, AX = XA$.We then write $X = A^\sharp$. It is well known that if $A^\sharp$ exists, it is unique,and the conclusion is given in[1].

On representations of the group inverse of block matrices,the authors have made efforts in [2]and [3]. Actually, Generalized inverses have wide applications in many areas such as special matrix theory, singular differential and difference equations and graph theory;see [4], [5] [6], [7] and [8].

Since the problem of finding an explicit representation for the Drazin (group) inverse of a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (where A and D are required to be square matrices) was proposed by Campbell and Meyer in 1979 ,considerable progress has been made .A condition for the existence of the group inverse of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given in [9] under the assumption that A and $I + CA^{-2}B$ are both invertible over any field;however,the representation of the group inverse is not given.And the representation of the group inverse of the block matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ over skew fields has been given in 2001 in [10]. Though the representation of the Drazin (group) inverse of the block matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ proposed as a problem by Campbell in 1983 in [11] (A is square, 0 is square null matrix) has not been given , there are some achievements about representations of the Drazin (group) inverse of the block matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ under special conditions. Some results are received on matrices over the complex field,e.g. in [12] when $A = B = I_n$ and in [13] when $A, B, C \in P, P^*, PP^*,\ P^2 = P$. Some results are over skew fields, e.g. in [14] when $A = I_n$ and $rank(CB)^2 = rank(B) = rank(C)$ and in [15] when $A = B, A^2 = A$. In addition, Group inverse of the product of two matrices,as well as some related properties over skew field are given in [16].

In this paper, we mainly give necessary and sufficient conditions for the existence and the representation of the group inverse of a block matrix $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$($A, B \in K^{n\times n}$,and $A^2 = A, B^2 = B$),similarly,we also reach a few conclusions under certain conditions.

2. Preliminaries

Lemma 2.1 Suppose $A \in K^{n\times n}$, then $A^\sharp$exists if and only if $rank(A) = rank(A^2)$.

Lemma 2.2 Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in K^{n\times n}, A \in k^{r\times r}$,then $M^\sharp$exists if and only if $A^\sharp, C^\sharp$ exists and rank(M)=rank(A)+rank(C),and then we have $M^\sharp = \begin{pmatrix} A^\sharp & X \\ 0 & C^\sharp \end{pmatrix}$,where $X = (A^\sharp)^2B(I - CC^\sharp) + (I - AA^\sharp)B(C^\sharp)^2 - A^\sharp BC^\sharp$.

Lemma2.3 Let $A^2 = A, B^2 = B$,and $rank(A - B) \leq rank[A(I - BA) + B(I - AB)]$,then $(A - B)^\sharp$exists.

Proof. One part, $rank[(A-B)^2] \geq rank[(A-B)^2(A+B)] = rank[A(I-BA)+B(I-AB)] \geq rank(A-B)$;And another part, $rank[(A-B)^2] \leq rank(A-B)$ apparently,then one can get $rank[(A-B)^2] = rank(A-B)$,then according to the conclusion of Lemma 2.1, $(A-B)^\sharp$exists.

Lemma 2.4 Let $A^2 = A, B^2 = B$,and $rank(A + B) \leq rank[A(I - BA) + B(I - AB)]$,then $(A + B)^\sharp$exists.

Proof. One part, $rank[(A+B)^2] \geq rank[(A+B)^2(A+B-2I)] = rank[A(I-BA) + B(I - AB)] \geq rank(A + B)$;And another part, $rank[(A + B)^2] \leq$

$rank(A+B)$ apparently,then one can get $rank[(A+B)^2] = rank(A+B)$,then according to the conclusion of Lemma 2.1, $(A+B)^\sharp$exists.

3. Conclusions

Theorem . suppose $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$,$(A, B \in K^{n\times n}, and A^2 = A, B^2 = B)$,then (i) $M^\sharp$ exists if and only if$rank(A-B) \leq rank[A(I-BA)+B(I-AB)]$.
(ii) if $M^\sharp$ *exists, and* $rank(A+B) \leq rank[A(I-BA)+B(I-AB)]$,

$$then \quad M^\sharp = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} (A-B)^\sharp & X \\ 0 & (A+B)^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}, \qquad (1)$$

where $X = (A-B)^{\sharp 2} B[I-(A+B)(A+B)^\sharp] + [I-(A-B)(A-B)^\sharp]B(A+B)^{\sharp 2} - (A-B)^\sharp B(A+B)^\sharp$.

Proof. (i)Proof of sufficient conditions.It is easy to prove that

$$rank(M) = rank\begin{pmatrix} A & B \\ B & A \end{pmatrix} = rank\begin{pmatrix} A-B & 0 \\ 0 & A+B \end{pmatrix}$$
$$= rank(A+B) + rank(A-B);$$
$$rank(M^2) = rank\begin{pmatrix} A^2+B^2 & AB+BA \\ AB+BA & A^2+B^2 \end{pmatrix}$$
$$= rank\begin{pmatrix} A+B & AB+BA \\ AB+BA & A+B \end{pmatrix}$$
$$= rank\begin{pmatrix} A+B & 0 \\ 0 & A+B-BAB-ABA \end{pmatrix}$$
$$= rank(A+B) + rank[A(I-BA)+B(I-AB)].$$

For the given condition $rank(A-B) \leq rank[A(I-BA)+B(I-AB)]$,we can get $rank(M) = rank(A+B) + rank(A-B) \leq rank(A+B) + rank[A(I-BA)+B(I-AB)] = rank(M^2)$;And with $rank(M) \geq rank(M^2)$,we can easily obtain $rank(M) = rank(M^2)$.Then according to the Lemma 2.1,$M^\sharp$ exists.
Proof of the necessary conditions .$M^\sharp$ exists,then $rank(M) = rank(M^2)$,and then $rank(A+B) + rank(A-B) = rank(A+B) + rank[A(I-BA)+B(I-AB)]$,so $rank[A(I-BA)+B(I-AB)] \geq rank(A-B)$ can be proved.
(ii) with the condition $rank(A-B) \leq rank[A(I-BA)+B(I-AB)]$ and the Lemma 2.3, $(A-B)^\sharp$ exists.Similarly,with the condition $rank(A+B) \leq rank[A(I-BA)+B(I-AB)]$ and the Lemma 2.4, $(A+B)^\sharp$ exists.

$$For \quad M = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A-B & B \\ 0 & A+B \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$
$$and \quad \begin{pmatrix} A-B & B \\ 0 & A+B \end{pmatrix} \rightarrow \begin{pmatrix} A-B & 0 \\ 0 & A+B \end{pmatrix},$$

we know $rank(M) = rank(A+B) + rank(A-B)$.And with the existence of $(A+B)^\sharp$ and $(A-B)^\sharp$,according to the Lemma 2.2,$M^\sharp$ has the form of (1).

Corollary 3.1 suppose $M = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$,($A \in K^{n\times n}$,and $A^2 = A$),then
$(i) \quad M^\sharp \quad exists;$

$$(ii) \quad if \quad M^\sharp \quad exists, then M^\sharp = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} 0 & A(2A)^{\sharp^2} \\ 0 & (2A)^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}.$$

Proof. We only need to replace B with A in Theorem,and easily prove that $(2A)^\sharp$ exists,then the conclusion comes true.

Corollary 3.2 suppose $M = \begin{pmatrix} A & I \\ I & A \end{pmatrix}$,($A \in K^{n\times n}$,and $A^2 = A$),then (i) $M^\sharp$ exists;$(ii) if \quad rank(A+I) \leq rank(A-I)$,then

$$M^\sharp = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} (A-I)^\sharp & X \\ 0 & (A+I)^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

where $X = (A-I)^{\sharp^2}[I-(A+I)(A+I)^\sharp] + [I-(A-I)(A-I)^\sharp](A+I)^{\sharp^2} - (A-I)^\sharp(A+I)^\sharp$.

Proof. In Theorem we replace B with I .For $rank(A-I) \leq rank[A(I-IA)+I(I-AI)$ and $rank(A+I) \leq rank(A-I)$,then $(A-I)^\sharp$ and $(A+I)^\sharp$ exist,so the Corollary 3.2 can be proved easily.

Corollary 3.3 suppose $M = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}$,($A \in K^{n\times n}$,and $A^2 = A$),then (i) $M^\sharp$ exists;(ii) if $M^\sharp$ exists, then

$$M^\sharp = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} (A-A^*)^\sharp & X \\ 0 & (A+A^*)^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

where $X = (A-A^*)^{\sharp^2}A^*[I-(A+A^*)(A+A^*)^\sharp]+[I-(A-A^*)(A-A^*)^\sharp]A^*(A+A^*)^{\sharp^2} - (A-A^*)^\sharp A^*(A+A^*)^\sharp$.

Proof. In Theorem we replace B with A^* .For $(A-A^*)$ and $(A+A^*)$ are respectively anti-Hermitian matrix and the Hermite matrix,they are unitarily similar to diagonal matrices,so $(A-A^*)^\sharp$ and $(A+A^*)^\sharp$ exist,then the Corollary 3.3 can be proved easily.

Corollary 3.4 suppose $M = \begin{pmatrix} A & AA^\sharp \\ AA^\sharp & A \end{pmatrix}$,($A \in K^{n\times n}$,and $A^2 = A$),then (i) $M^\sharp$ exists;(ii)if $rank(A+AA^\sharp) \leq rank(A-AA^\sharp)$, then $M^\sharp = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} (A-AA^\sharp)^\sharp & X \\ 0 & (A+AA^\sharp)^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$,

where $X = (A-AA^\sharp)^{\sharp^2}[AA^\sharp-(A+AA^\sharp)(A+AA^\sharp)^\sharp] + [I-(A-AA^\sharp)(A-AA^\sharp)^\sharp]AA^\sharp(A+AA^\sharp)^{\sharp^2} - (A-AA^\sharp)^\sharp AA^\sharp(A+AA^\sharp)^\sharp$.

Proof. In Theorem we replace B with $AA^\sharp$ .For $rank(A-AA^\sharp) = ran[(A(I-AA^\sharp A)+AA^\sharp(I-AA^\sharp A)]$ and $rank(A+AA^\sharp) \leq rank(A-AA^\sharp)$,so $(A-AA^\sharp)^\sharp$ and $(A+AA^\sharp)^\sharp$ exists.Therefore it ,then the Corollary 3.4 can be proved easily.

References

[1] Wa-Jin Zhuang,Involutory functions and generalized inverses of matrices over an arbitrary skew fields,Northeast,Math[J], (Chinese) 1:57-65, 1987.

[2] G. Wang e.t,Generalized Inverse Theory and Computations,Science Press,Beijing, 2003.

[3] K.P.S. Bhaskara Rao,The Theory of Generalized Inverses over Commutative Rings,Taylor-Francis, New York, 2002.

[4] S.L. Campbell and C.D. Meyer, Generalized Inverses of Linear Transformations,Dover, New York, 1991.

[5] M. Catral, D.D. Olesky, P. van den Driessche,Group inverses of matrices with path graphs,Electron[J] Linear Algebra, 17:219-233, 2008.

[6] G.H. Golub , C. Greif,On solving block-structured indefinite linear systems,SIAM [J] Sci.Comput., 24:2076-2092, 2003.

[7] I.C.F. Ipsen,A note on preconditioning nonsymmetric matrices,SIAM [J],Sci. Comput.,23:1050-1051, 2001.

[8] Y. Wei ,H. Diao,On group inverse of singular Toeplitz matrices,Linear Algebra Appl.,399:109-123, 2005.

[9] X. Chen ,R.E. Hartwig,Group inverse of a triangular matrix,Linear Algebra Appl.,237/238:97-108, 1996.

[10] C. Cao,Some results of group inverses for partitioned matrices over skew fields,Heilongjiang University Natural Science Journal (Chinese), 18:5-7, 2001.

[11] S.L. Campbell,The Drazin inverse and systems of second order linear differential equations,Linear Multi-linear Algebra, 14:195-198, 1983.

[12] N. Castro-Gonzales ,Representations of the Drazin inverse for a class of block matrices,Linear Algebra Appl., 400:253-269, 2005.

[13] C. Cao ,X. Tang,Representations of the group inverse of some 2×2 block matrices,Int.Math. Forum, 31:1511-1517, 2006.

[14] C. Bu,On group inverses of block matrices over skew fields[J] Math., 35:49C52, 2002.

[15] C. Bu, J. Zhao,J. Zheng,Group inverse for a class 2×2 block matrices over skew fields,Appl. Math. Comput., 204:45-49, 2008.

[16] C. Bu , C. Cao,Group inverse of product of two matrices over skew field[J] ,Math Study,35:43-438, 2002.

[17] Wajin Zhuang , Guidance of matrix theory on skew field[M], Science press,Beijing(2006).

Received: February, 2012