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On Apéry Sets of Symmetric Numerical Semigroups with $e(S) = p$

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Abstract

The concepts of an Apéry set is important role in numerical semigroup theory. In this paper we characterizes Apéry set of numerical semigroup and generalizes Apéry set of symmetric numerical semigroup with $e(S) = p$, where p is a positive integer.

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1 Introduction

A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and generates $\mathbb{Z}$ as a group. It is well known that every numerical semigroup is finitely generated (see [2]), i.e. $S =< x_1, x_2, \ldots, x_k >$ for some $x_1, x_2, \ldots, x_k \in S$ and $k \in \mathbb{N} \setminus \{0\}$ such that

$$< x_1, x_2, \ldots, x_k >= \{\sum_{i=1}^{k} n_i x_i \mid n_i \in \mathbb{N}\}.$$

In [1], it was shown that:

$$G.C.D.(x_1, x_2, \ldots, x_k) = 1 \Leftrightarrow Card(\mathbb{N} \setminus S) < \infty.$$

For a numerical semigroup S we define the following:

$$g(S) := max\{x \in \mathbb{Z} \mid x \notin S\};$$

$$n(S) := Card(\{0, 1, \ldots, g(S)\} \cap S);$$

$$H(S) := \{x \in \mathbb{N} \min x \notin S\}.$$

And $g(S), n(S)$ and $H(S)$ are called Frobenius number, genus and gap of S, respectively. If a numerical semigroup $S =< x_1, x_2, \ldots, x_k >$, then $< x_1, x_2, \ldots, x_k >$ is called system of generators of S and will be called minimal system of generators of S if no proper subset of $\{x_1, x_2, \ldots, x_k\}$ generates S. If a numerical semigroup has a unique minimal system of generators $\{x_1, x_2, \ldots, x_k\}$ which is $x_1 < x_2 < \ldots < x_k$. Then the number x_1 and k are called the multiplicity and the embedding dimension of S, respectively. The embedding dimension of S is denoted by $e(S)$. We say that a numerical semigroup S is symmetric if $g(S) - x \in S$ for all $x \in \mathbb{Z} \setminus S$. For $n \in S \setminus \{0\}$, we define the Apéry set of the element n as the set of all the least elements in S congruent with i modulo n and denoted by $Ap(S, n)$ and it can be proved that $Ap(S, n) = \{x \in S \mid x - n \notin S\}$. Moreover, it is clearly that, $Card(Ap(S, n)) = n$ and $g(S) = max(Ap(S, n)) - n$ (see [5]).

2 Apéry Sets of Symmetric Numerical Semigroups with $e(S) = p$

Let S be a numerical semigroup with $e(S) = p$, i.e., $S =< n_1, n_2, \ldots, n_p >$ for some $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$. For each $i \in \{1, 2, \ldots, p\}$ we define a set
$D_{S,n_i} := \{x \in S \mid x = an_i + b \; for\; some\; a \in \mathbb{N}^+ \;and\; b \in < n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p > \}$. Then we obtains

Theorem 2.1. *Let S is a numerical semigroup and $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$. If $S =< n_1, n_2, \ldots, n_p >$, then $Ap(S, n_i) = S \setminus D_{S,n_i}$ for all $i = 1, 2, \ldots, p$.*

Proof. Assume that $x \in Ap(S, n_i)$, we get that $x - n_i \notin S$. Suppose that $x \notin S \setminus D_{S,n_i}$, then $x \in D_{S,n_i}$. By definition of D_{S,n_i}, there exist $a \in \mathbb{N}^+$ and $b \in< n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p >$ such that $x = an_i + b$. Therefore we have that
$x - n_i = (an_i + b) - n_i = (a - 1)n_i + b \in S$. Which is contradicts to $x - n_i \notin S$. Hence $x \in S \setminus D_{S,n_i}$. This means that $Ap(S, n_i) \subseteq S \setminus D_{S,n_i}$.
Conversely, assume that $x \in S \setminus D_{S,n_i}$, then $x \notin D_{S,n_i}$, i.e., $x \neq an_i + b$ for all $a \in \mathbb{N}^+$ and $b \in< n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p >$. Suppose that $x \notin Ap(S, n_i)$, then $x - n_i \in S$. Therefore there exist $a_1, a_2, \ldots, a_p \in \mathbb{N}$ such that $x - n_i = a_1n_1 + a_2n_2 + \ldots + a_pn_p$. So $x = (a_1 + 1)n_1 + a_2n_2 + \ldots + a_pn_p \in D_{S,n_i}$. Which is contradicts to $x \in S \setminus D_{S,n_i}$. Hence $x \in Ap(S, n_i)$. This implies that $S \setminus D_{S,n_i} \subseteq Ap(S, n_i)$. There follows we get that $Ap(S, n_i) = S \setminus D_{S,n_i}$. □

For the particular case $e(S) = 2$, i.e., $S =< n_1, n_2 >$ for some $n_1, n_2 \in \mathbb{N}^+$, we have that $D_{S,n_1} = \{an_1 + bn_2 \mid a \in \mathbb{N}^+, b \in \mathbb{N}\}$ and $D_{S,n_2} = \{an_1 + bn_2 \mid a \in \mathbb{N}, b \in \mathbb{N}^+\}$. Then we obtains

Corollary 2.2. *Let S is a numerical semigroup and $n_1, n_2 \in \mathbb{N}^+$. If $S =< n_1, n_2 >$, then $Ap(S, n_1) = \{0, n_2, 2n_2, \ldots, (n_1 - 1)n_2\}$ and $Ap(S, n_2) = \{0, n_1, 2n_1, \ldots, (n_2 - 1)n_1\}$.*

Proof. By Theorem 2.1, we get that

$$\begin{aligned} Ap(S, n_1) &= S \setminus D_{S,n_1} \\ &= < n_1, n_2 > \setminus \{an_1 + bn_2 \mid a \in \mathbb{N}^+, b \in \mathbb{N}\} \\ &= \{xn_1 + yn_2 \mid x, y \in \mathbb{N}\} \setminus \{an_1 + bn_2 \mid a \in \mathbb{N}^+, b \in \mathbb{N}\} \\ &= \{0, n_2, 2n_2, \ldots, (n_1 - 1)n_2\}. \end{aligned}$$

Similarly, one can prove that $Ap(S, n_2) = \{0, n_1, 2n_1, \ldots, (n_2 - 1)n_1\}$.

□

Since we have that every symmetric numerical semigroups are numerical semigroups. Therefore the results are also true for symmetric numerical semigroup.

Theorem 2.3. *Let S is a symmetric numerical semigroup and $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$. If $S =< n_1, n_2, \ldots, n_p >$, then $Ap(S, n_i) = S \setminus D_{S,n_i}$ for all $i = 1, 2, \ldots, p$.*

Corollary 2.4. *(see e.g. [3]) Let S is a symmetric numerical semigroup and $n_1, n_2 \in \mathbb{N}^+$. If $S =< n_1, n_2 >$, then $Ap(S, n_1) = \{0, n_2, 2n_2, \ldots, (n_1 - 1)n_2\}$ and $Ap(S, n_2) = \{0, n_1, 2n_1, \ldots, (n_2 - 1)n_1\}$.*

Theorem 2.5. *Let S be a numerical semigroup and $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$. If $S =< n_1, n_2, \ldots, n_p >$ and there exists $k \in \mathbb{N}^+$ such that $H =< n_1, \ldots, n_{i-1}, kn_i, n_{i+1}, \ldots, n_p >$ is a numerical semigroup for some $i \in \{1, 2, \ldots, p\}$, then H is a subsemigroup of S and $Ap(S, n_i) \subseteq Ap(H, kn_i)$.*

Proof. Clearly, H is a subsemigroup of S. Let $x \in Ap(S, n_i)$. By Theorem 2.1, we get that $x \in S \setminus D_{S,n_i}$, i.e., $x \in S$ and $x \neq an_i + b$ for all $a \in \mathbb{N}^+$ and $b \in < n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p >$. Then $x \in < n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p >$ and $x \neq c(kn_i) + b$ for all $c \in \mathbb{N}^+$ and $b \in < n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p >$. Thus $x \in < n_1, \ldots, n_{i-1}, kn_i, n_{i+1}, \ldots, n_p >$ and $x \neq c(kn_i) + b$ for all $c \in \mathbb{N}^+$. Therefore $x \in H$ and $x \notin D_{H,kn_i}$. This implies that $x \in H \setminus D_{H,kn_i}$. Hence $Ap(S, n_i) \subseteq Ap(H, kn_i)$.

□

For a symmetric numerical semigroup we obtains

Corollary 2.6. *Let S be a symmetric numerical semigroup and $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$. If $S =< n_1, n_2, \ldots, n_p >$ and for each $i \in \{1, 2, \ldots, p\}$ there exists $k \in \mathbb{N}^+$ such that $H =< n_1, \ldots, n_{i-1}, n_{n+1}, \ldots, n_p >$ is a symmetric numerical semigroup, then H is a subsemigroup of S and $Ap(S, n_i) \subseteq Ap(H, kn_i)$.*

Let S be a numerical semigroup with Frobenius number $g(S)$ and genus $n(S)$. We have that, if S has an embedding dimension two, then $n(S) = (g(S) + 1)/2$ (see [4]). But its not true in general. For symmetric numerical semigroups we have the theorem

Theorem 2.7. *If S is a symmetric numerical semigroup with $e(S) = p$, then $n(S) = (g(S) + 1)/2$.*

Proof. Let A and B be subsets of S such that $A := \{x \in S \mid 0 \leq x < g(S)\}$ and $B := \{y \notin S \mid 0 < y \leq g(S)\}$. Clearly, $\{0, 1, \ldots, g(S)\} = A \cup B$ and $A \cap B = \emptyset$. Since S is a symmetric numerical semigroup, then a mapping $\varphi : A \to B$ such that $\varphi(x) = y$ iff $g(S) - y = x$ is bijective. Thus

$$\begin{aligned} g(S)+1 &= Card\{0,1,\ldots,g(S)\} \\ &= Card(A \cup B) \\ &= Card(A) + Card(B) \\ &= 2Card(A) \\ &= 2n(S). \end{aligned}$$

Therefore we get that $n(S) = (g(S) + 1)/2$. □

If S is a symmetric numerical semigroup and from Theorem 2.7, we have that the set B is the set of all gaps of S which is denoted by $H(S)$, i.e., $H(S) = \{x \mid x \in \mathbb{N} \setminus S\}$. Then $Card(H(S)) = Card(B) = Card(A) = n(S)$. The set of all gaps of S can be written in the form of Apéry set as in [3], but Theorem 3 in P. 483 is not true. For example $S =< 5, 7 >$ is a symmetric numerical semigroup, but $H(S) \neq Ap(S, 5) \cup Ap(S, 7) \cup \{12\}$. Then we obtains

Theorem 2.8. *Let S be a numerical semigroup and $e(S) = p$. If $S =< n_1, n_2, \ldots, n_p >$ for some $n_1, n_2, \ldots, n_p \in \mathbb{N}^+$, then $H(S) = \{0, 1, \ldots, g(S)\} \setminus (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$.*

Proof. Let $h \in H(S)$. Then $h \in \{0, 1, \ldots, g(S)\}$ and $h \notin S$. Therefore $h \in \{0, 1, \ldots, g(S)\}$ and $h \notin (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$. This implies that $H(S) \subseteq \{0, 1, \ldots, g(S)\} \setminus (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$.

Conversely, let $x \in \{0, 1, \ldots, g(S)\} \setminus (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$. Then $x \in \{0, 1, \ldots, g(S)\}$ and $x \notin (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$. Since $x \notin (\bigcup_{i=1}^{p} (Ap(S, n_i) \cup D_{S,n_i})$, then

$x \neq 0$. Suppose that $x \in S$. Then $x = a_1n_1 + a_2n_2 + \ldots + a_pn_p$ for some $a_1, a_2, \ldots, a_p \in \mathbb{N}$. Since $x \neq 0$, then there exists $a_j \in \mathbb{N}^+$ for some $j \in \{1, 2, \ldots, p\}$ such $x = a_jn_j + b$ for some $b \in < n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_p >$. This implies that $x \in D_{S,n_j}$. This is contradicts to $x \notin (\bigcup_{i=1}^{p}(Ap(S, n_i) \cup D_{S,n_i})$. So $x \notin S$. Therefore $x \in \{0, 1, \ldots, g(S)\} \setminus S$. Since $\{0, 1, \ldots, g(S)\} \subseteq \mathbb{N}$, we get that $x \in \mathbb{N} \setminus S$.

□

In the particular case, $e(S) = 2$. Since $Ap(S, n_1) = \{0, n_2, 2n_2, \ldots, (n_1 - 1)n_2)\}$, $Ap(S, n_2) = \{0, n_1, 2n_1, \ldots, (n_2 - 1)n_1)\}$, $D_{S,n_1} = \{x \in S \mid x = an_1 + b \textit{ for some } a \in \mathbb{N}^+ \textit{ and } b \in < n_2 >\}$ and $D_{S,n_2} = \{x \in S \mid x = an_2 + b \textit{ for some } a \in \mathbb{N}^+ \textit{ and } b \in < n_1 >\}$ and it not difficult to prove that $Ap(S, n_1) \cup Ap(S, n_2) \cup D_{S,n_1} \cup D_{S,n_2} = Ap(S, n_1) \cup Ap(S, n_2) \cup \{x \in S \mid x = an_1 + bn_2 \textit{ for some } a, b \in \mathbb{N}^+\}$. Then we obtains

Corollary 2.9. *Let S be a numerical semigroup and $e(S) = 2$. If $S =< n_1, n_2 >$ for some $n_1, n_2 \in \mathbb{N}^+$, then $H(S) = \{0, 1, \ldots, g(S)\} \setminus (Ap(S, n_1) \cup Ap(S, n_2) \cup \{x \in S \mid x = an_1 + bn_2 \textit{ for some } a, b \in \mathbb{N}^+\})$.*

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