International Mathematical Forum, Vol. 7, 2012, no. 37, 1799 - 1804

On Binumerical Semigroups

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Abstract

The concepts of numerical semigroups plays an important role in the theory of semigroups. In this paper we introduce the basic concepts of binumerical semigroups which build up from numerical semigroups.

Mathematics Subject Classification: 20M14, 20F50

Keywords: Numerical Semigroups, Binumerical Semigroups

1 Introduction

A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and generates $\mathbb{Z}$ as a group. It is well known that every numerical semigroup is finitely generated (see [2]), i.e. $S =< x_1, x_2, \dots, x_k >$ for some $x_1, x_2, \dots, x_k \in S$ and $k \in \mathbb{N} \setminus \{0\}$ such that

$$< x_1, x_2, \dots, x_k >= \{\sum_{i=1}^{k} n_i x_i \mid n_i \in \mathbb{N}\}.$$

In [1], it was shown that:

$$G.C.D.(x_1, x_2, \dots, x_k) = 1 \Leftrightarrow Card(\mathbb{N} \setminus S) < \infty.$$

For a numerical semigroup S we define the following:

$$g(S) := max\{x \in \mathbb{Z} \mid x \notin S\};$$

$$n(S) := Card(\{0, 1, \dots, g(S)\} \cap S);$$

$$H(S) := \{x \in \mathbb{N} \mid x \notin S\}.$$

And $g(S), n(S)$ and $H(S)$ are called Frobenius number, genus and gap of S, respectively. If a numerical semigroup $S =< x_1, x_2, \dots, x_k >$, then $< x_1, x_2, \dots, x_k >$ is called system of generators of S and will be called minimal system of generators of S if no proper subset of $\{x_1, x_2, \dots, x_k\}$ generates S. If a numerical semigroup has a unique minimal system of generators $\{x_1, x_2, \dots, x_k\}$ which is $x_1 < x_2 < \dots < x_k$. Then the number x_1 and k are called the multiplicity and the embedding dimension of S, respectively. The embedding dimension of S is denoted by $e(S)$. We say that a numerical semigroup S is symmetric if $g(S) - x \in S$ for all $x \in \mathbb{Z} \setminus S$. For $n \in S \setminus \{0\}$, we define the Apéry set of the element n as the set of all the least elements in S congruent with i modulo n and denoted by $Ap(S, n)$ and it can be proved that $Ap(S, n) = \{x \in S \mid x - n \notin S\}$. Moreover, it is clearly that, $Card(Ap(S, n)) = n$ and $g(S) = max(Ap(S, n)) - n$ (see [3]).

2 Binumerical Semigroups

At first, we define the binary operation $+$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) + (c, d) = (a + c, b + d)$, for all $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$.

Definition 2.1. *Let S be a subset of $\mathbb{N} \times \mathbb{N}$, S is a binumerical semigroup if S is a subsemigroup of $(\mathbb{N} \times \mathbb{N}, +)$ with $(0, 0) \in S$ and S generates $\mathbb{Z} \times \mathbb{Z}$ as a group.*

It is clearly that $S_1 \times S_2$ is binumerical semigroup for every numerical semigroups S_1 and S_2. Nevertheless binumerical semigroup need not to be product of two numerical semigroups, for instance $S = \{(0, 0), (a, b), (a+1, b+1), \dots\}$. In the case of numerical semigroup we have that every numerical semigroup has finite complement in $\mathbb{N}$. In the case of binumerical semigroups we obtains

Theorem 2.2. *If S is a binumerical semigroup, then $\{\{x\} \times \mathbb{N} \mid (x, n) \notin S \ for\ all\ n \in \mathbb{N}\}$ and $\{\mathbb{N} \times \{y\} \mid (m, y) \notin S \ for\ all\ m \in \mathbb{N}\}$ are finite.*

Proof. Let S be a binumerical semigroup. Suppose that $\{\{x\} \times \mathbb{N} \mid (x, n) \notin S \ for\ all\ n \in \mathbb{N}\}$ is an infinite set, then $\{x \in \mathbb{N} \mid (x, n) \notin S \ for\ all\ n \in S\}$ is an infinite, so $\mathbb{N} \setminus \{a \in \mathbb{N} \mid \exists y \in \mathbb{N}, (a, y) \in S\}$ is an infinite set. This implies that $\{a \in \mathbb{N} \mid \exists y \in \mathbb{N}, (a, y) \in S\}$ is not a numerical semigroup, it can not be generates $\mathbb{Z}$ as a group. This contradicts with S generates $\mathbb{Z} \times \mathbb{Z}$. Therefore, $\{\{x\} \times \mathbb{N} \mid (x, n) \notin S \ for\ all\ n \in \mathbb{N}\}$ is finite.
Similarly, $\{\mathbb{N} \times \{y\} \mid (m, y) \notin S \ for\ all\ m \in \mathbb{N}\}$ is finite. □

If S is a binumerical semigroup, we will form the subsets of $\mathbb{N}$, $D(S) = \{a \in \mathbb{N} \mid \exists n \in \mathbb{N}, (a, n) \in S\}$ and $R(S) = \{b \in \mathbb{N} \mid \exists m \in \mathbb{N}, (m, b) \in$

$S\}$. It easy to see that $D(S)$ and $R(S)$ are smallest pair of numerical semigroups with $S \subseteq D(S) \times R(S)$. We know that every numerical semigroup is finitely generated, but not true for the case of binumerical semigroup, for instance, any binumerical semigroups which are not $\mathbb{N} \times \mathbb{N}$ and contain $\{(2,4),(3,5),(2,6),(4,7),(2,8),(5,9),\dots\}$ are not finitely generated. Nevertheless, there are many binumerical semigroups which are finitely generated. For example $\mathbb{N} \times \mathbb{N}$ and $S_1 \times S_2$, where S_1 and S_2 are binumerical semigroups. For generating system of a subset of $\mathbb{N} \times \mathbb{N}$ we define

Definition 2.3. *Let A be a subset of $\mathbb{N} \times \mathbb{N}$. A subsemigroup of $\mathbb{N} \times \mathbb{N}$ generated by A is denoted by $< A >$ and defined by $< A >:= \{(\sum_{i=1}^{n} \alpha_i a_i, \sum_{i=1}^{n} \alpha_i b_i) \mid \forall \alpha_i \in \mathbb{N}, (a_i, b_i) \in A\}$. If S is a binumerical semigroup and $S =< A >$, then A is called a generator of S and if A is finite, then S is called finitely generated.*

We obtains

Theorem 2.4. *A binumerical semigroup S is finitely generated if and only if the following sets $\{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\} \cup \{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\}$ are finite, where $D(S) =< a_1, \dots, a_n >$ and $R(S) =< b_1, \dots, b_m >$.*

Proof. Let S be a binumerical semigroup and $D(S) =< a_1, \dots, a_n >$ and $R(S) =< b_1, \dots, b_m >$. Since S is finitely generated and any element $a \in \mathbb{N}$ such that $(a,b) \in S$ for some $b \in \mathbb{N}$ is a finite sum of $a_1, a_2, \dots, a_n$. Therefore the set $\{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\}$ is finite. Similarly, $\{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\}$ is finite. Conversely, we have show that
$S =< \{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\} \cup \{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\} >$. Clearly that $< \{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\} \cup \{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\} > \subseteq S$. Let $(a,b) \in S$, we get that $a \in D(S)$ and $b \in R(S)$. Then $a \in< a_1, \dots, a_n >$ and $b \in< b_1, \dots, b_m >$. This implies that $(a,b) \in< \{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\} \cup \{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\} >$. Therefore, $S =< \{(a_i, \sum_{j=1}^{m} \beta_j b_j) \in S \mid \exists \beta_j \in \mathbb{N}\} \cup \{(\sum_{i=1}^{n} \alpha_i a_i, b_j) \in S \mid \exists \alpha_i \in \mathbb{N}\} >$.

□

For a subsystem of binumerical semigroup, we define

Definition 2.5. *Let S be a binumerical semigroup. A subset $H \subseteq \mathbb{N} \times \mathbb{N}$ is called a subbinumerical semigroup of S if $H \subseteq S$ and H is a binumerical semigroup.*

It is clearly that, every binumerical semigroup S is a subbinumerical semigroup of S, and the intersections of subbinumerical semigroup is again binumerical semigroup. Therefore, the class $Sub(S)$ of all subbinumerical semigroups of S forms a complete lattice, where the operations $\bigwedge$ and $\bigvee$ are defined by $H_1 \bigwedge H_2 := H_1 \bigcap H_2$ and $H_1 \bigvee H_2 :=< H_1 \bigcup H_2 >$, respectively. Moreover, we have the relation between numerical semigroup and binumerical semigroup.

Theorem 2.6. *If S_1 is a numerical semigroup, then S_1 is isomorphic to binumerical semigroup S for some binumerical S.*

Proof. Let S_1 be a numerical semigroup and $\{a_1, \dots, a_n\}$ be the generating set of S_1. Let S be a binumerical semigroup which is generated by $\{(a_1, b_1), \dots, (a_n, b_n)\}$ and we define a mapping $f : S_1 \to S$ by $a_i \mapsto (a_i, b_i)$, it easy to see that f is bijective. Let $a, b \in S_1$, then $a = \alpha_1 a_1 + \ldots + \alpha_n a_n$ and $b = \beta_1 a_1 + \ldots + \beta_n a_n$ for some $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{N}$. Therefore,

$$\begin{aligned} f(a+b) &= f((\alpha_1 a_1 + \ldots + \alpha_n a_n) + (\beta_1 a_1 + \ldots + \beta_n a_n)) \\ &= f((\alpha_1 + \beta_1) a_1 + \ldots + (\alpha_n + \beta_n) a_n) \\ &= ((\alpha_1 + \beta_1) a_1 + \ldots + (\alpha_n + \beta_n) a_n, (\alpha_1 + \beta_1) b_1 + \ldots + (\alpha_n + \beta_n) b_n) \\ &= (\alpha_1 a_1 + \ldots + \alpha_n a_n, \alpha_1 b_1 + \ldots + \alpha_n b_n) + \\ &\quad (\beta_1 a_1 + \ldots + \beta_n a_n, \beta_1 b_1 + \ldots + \beta_n b_n) \\ &= f(a) + f(b) \end{aligned}$$

Hence, $S_1 \cong S$.

□

The converse of the above Theorem is not true, for instance a binumerical semigroup S contains the set $\{(2, 4), (3, 5), (2, 6), (4, 7), (2, 8), (5, 9), \ldots\}$ is not isomorphic to any numerical semigroup. The homomorphism between two of binumerical semigroups is define as usual semigroup homomorphism.

Definition 2.7. *Let S and H be the binumerical semigroups. A mapping $f : S \to H$ is call a homomorphism if and only if $f(a + b) = f(a) + f(b)$ for all $a, b \in S$.*

It easy to see that, every zero function and identity function are homomorphisms. We need the following condition for the characterization of homomorphisms of binumerical semigroups.

Lemma 2.8. *A mapping f is a homomorphism of numerical semigroups if and only if f is either zero or embedding.*

Proof. Clearly, zero and embedding mappings are homomorphisms. Assume that $f : S \to H$ is a homomorphism of numerical semigroups and that f is not zero. Suppose that f is not an embedding, then there is $m \in S$ and $n \in H$ such that $f(m) = n$ and $m \neq n$. Suppose that there is $k \in S$ such that

$f(k) = k$. Since f is a homomorphism, we get that $f(km) = kf(m) = kn$ and $f(km) = mf(k) = mk$. Then $m = n$, this contradicts with $m \neq n$. Therefore, $f(k) \neq k$ for all $k \in S$. Since f is homomorphism, then $af(b) = bf(a)$ for all $a, b \in S$. This equation implies that $f(a) = 0 = f(b)$ for all $a, b \in S$. Then we get a contradiction, because of f is not zero. Hence f is embedding.

□

Lemma 2.9. *If $f : S \to H$ is a homomorphism of binumerical semigroups, then the mappings $f_1 : D(S) \to D(H)$ and $f_2 : R(S) \to R(H)$ which are defined by $f_1(a) = c$ and $f_2(b) = d$, where $c \in D(H), d \in R(H)$ and $f(a, b) = (c, d)$ are homomorphisms.*

Proof. For each $a \in D(S)$, we have a subset $[a] := \{x \in \mathbb{N} \mid (a, x) \in S\}$ of $R(S)$ is not empty. Let $a_1, a_2 \in D(S)$ such that $a_1 = a_2$. Then $[a_1] = [a_2]$, we let $d' \in [a_1] = [a_2]$ such that $(a_1, d'), (a_2, d') \in S$. Therefore, $f(a_1, d') = f(a_2, d')$. By definition, we get that $f(a_1) = f(a_2)$. Let $x, y \in D(S)$, then there are $x', y' \in R(S)$ such that $(x, x'), (y, y') \in S$. Since f is homomorphism, then $f(x + y, x' + y') = f((x, x') + (y, y')) = f(x, x') + f(y, y')$. This implies that $f_1(x + y) = f_1(x) + f_1(y)$. Then f_1 is a homomorphism. Similarly, f_2 is also a homomorphism.

□

Then we get the Theorem

Theorem 2.10. *A mapping f is a homomorphism of binumerical semigroups if and only if f is either zero or embedding.*

Proof. Clearly, zero and embedding mappings are homomorphisms. Let $f : S \to H$ be a homomorphism of binumerical semigroups and f is not zero. Let $(a, b) \in S$. Since f is a homomorphism, then there are homomorphisms $f_1 : D(S) \to D(H)$ and $f_2 : R(S) \to R(H)$ such that $f(a, b) = (f_1(a), f_2(b))$. By Lemma 2.1, we get that f_1 and f_2 are embeddings, therefore $f(a, b) = (a, b)$. Hence f is an embedding.

□

As in the case of algebra, one can define that a congruence on an algebra $\mathcal{A}$ is a kernel of some homomorphism.

Definition 2.11. *An equivalence relation θ on a binumerical semigroup S is called congruence on S if θ is equal to the kernel of homomorphism from S to H, for some a binumerical semigroup H.*

Corollary 2.12. *If S is a binumerical semigroup, then the congruences on S is either $S \times S$ or Δ_S.*

Proof. By Theorem 2.4, a homomorphism from S to H such that H is a binumerical semigroup is either zero or embedding. Then the kernel of such homomorphisms are $S \times S$ and Δ_S.

□

ACKNOWLEDGEMENTS. This research was supported by the Faculty of Science of Mahasarakham University Thailand.

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Received: February, 2012