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Enumerating Binary Strings without r-Runs of Ones

M. A. Nyblom

School of Mathematics and Geospatial Science
RMIT University
Melbourne, Victoria, Australia
michael.nyblom@rmit.edu.au

Abstract

The number of binary strings of length n containing no substrings consisting of r consecutive ones is examined and shown to be given in terms of a well known integer sequence namely, the r-Fibonacci sequence. In addition, difference equations for the number of zeros and the total number of runs within these binary strings are derived.

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1 Introduction

The sequence of Fibonacci numbers is arguably the most well known and studied sequence in discrete mathematics. One remarkable feature of this sequence and its varied generalizations is the way in which they frequently occur in connection with problems of enumeration. As an illustration, consider the problem of counting the number of binary strings that is, a finite sequence of zeroes and ones of length n, in which there are no pairs of consecutive 1's. It is well known (see [1],[4, Example 10.10]) that the number of such binary strings is F_{n+2}, where F_n denotes the n-th Fibonacci number generated from the difference equation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$.

Thus for binary strings of length, say $n = 3$, there are exactly $F_5 = 5$ binary strings in which there are no pairs of consecutive 1's, namely $000, 001, 010, 100$ and 101. A similar result for ternary strings that is, a finite sequence consisting of terms from the alphabet $\{0, 1, 2\}$ was established in [3], where the number of ternary strings of length $n \geq 2$, having no pairs of consecutive ones was shown to be equal to the n-th term of the sequence $\{a_n\}$, generated from the difference equation $a_n = 2a_{n-1} + 2a_{n-2}$, for $n \geq 2$, with $a_0 = 1$ and $a_1 = 3$.
The property of a binary string having no pairs of consecutive 1's, (or equivalently no pairs of consecutive zeros), can easily be generalized to the property that a binary string has no substrings of length r consisting of consecutive r ones, where r is a fixed integer greater than or equal to 2. We shall refer to this property in short, by saying that a binary string has no r-runs of 1's. Not surprisingly, the Fibonacci numbers are again intimately connected with the problem of enumerating such binary strings, as illustrated by R. Grimaldi and S. Heubach, who established in [2] that the total number of binary strings of length n having no odd numbered runs of 1's is given by F_{n+1}. In this paper, we shall investigate the related problem of enumerating, for any fixed integer $r \geq 2$, the number of binary strings of length n having no r-runs of 1's. As in [2] our main result. proved in Section 2, will be to show that the number of such binary strings is given in terms of a well known integer sequence, namely U_{n+r}, where $\{U_n\}$ denoted the r-Fibonacci sequence generated by the r-th order linear difference equation

$$U_n = \sum_{i=1}^{r} U_{n-i} \ ,$$

for $n \geq r$, with $U_0 = U_1 = \cdots = U_{r-2} = 0$ and $U_{r-1} = 1$. This result clearly encompasses the one quoted above concerning the counting of binary strings having no pairs of consecutive 1's, since for $r = 2$ we have $U_{n+r} = F_{n+2}$. In addition, we shall in Section 3 establish difference equations used in the calculations of various characteristics of the binary sequences in question, such as the total number of zeros and the total number of runs that is, the number of maximal length substrings consisting either entirely of zeros or entirely of ones.

2 Counting Binary Strings Without r-runs of Ones

In what follows suppose for integers $n \geq r \geq 2$, we let $T_r(n)$ denote the number of binary strings of length n having no r-runs of 1's. Our first task here will be to demonstrate that one can recursively calculate $T_r(n)$ from an r-th order homogeneous difference equation, with r initial conditions $T_r(1), T_r(2), \ldots, T_r(r)$ to be defined as follows.

Theorem 2.1 *For a fixed integer $r \geq 2$ the number of binary strings of length n having no r-runs of 1's, satisfies the following r-th order linear difference equation*

$$T_r(n) = \sum_{i=1}^{r} T_r(n-i) \ , \tag{1}$$

for $n > r$, with the r initial conditions $T_r(s) = 2^s$, for $s = 1, \ldots, r-1$ and $T_r(r) = 2^r - 1$.

Proof: Suppose $S_r(n)$ denotes the set of binary strings of length $n > r$ having the property that there are no r-runs of 1's. Then $S_r(n) = A_r(n) \cup B_r(n)$, where $A_r(n)$ and $B_r(n)$ denote the disjoint sets containing those binary strings whose left hand entry contains a 0 and a 1 respectively. Now, if $f_{1,r}(n) = |A_r(n)|$ and $f_{2,r}(n) = |B_r(n)|$, then the total number of binary strings in $S_r(n)$ is given by $T_r(n) = f_{1,r}(n) + f_{2,r}(n)$. We first construct the recurrence relation for $T_r(n)$, where $n > r$. Considering an arbitrary binary string $x \in A_r(n)$, observe that x can uniquely be written as the concatenation $0y$, where $y \in S_r(n-1) = A_r(n-1) \cup B_r(n-1)$. Thus as $A_r(n-1) \cap B_r(n-1) = \emptyset$ we deduce that

$$f_{1,r}(n) = f_{1,r}(n-1) + f_{2,r}(n-1) \ . \tag{2}$$

Alternatively, for an arbitrary binary string $x \in B_r(n)$, further note that x can be uniquely written as one and only one of the following $r-1$ concatenations $\underbrace{11\cdots1}_{i1's}y$, where $y \in A_r(n-i)$, for $i = 1, 2, \ldots, r-1$, that is

$$B_r(n) = \bigcup_{i=1}^{r-1} \{\underbrace{11\cdots1}_{i1's}y : y \in A_r(n-i)\} \ .$$

Consequently as $\{\underbrace{11\cdots1}_{i1's}y : y \in A_r(n-i)\} \cap \{\underbrace{11\cdots1}_{j1's}y : y \in A_r(n-j)\} = \emptyset$, for $i \neq j$, observe that

$$f_{2,r}(n) = \left|\bigcup_{i=1}^{r-1}\{\underbrace{11\cdots1}_{i1's}y : y \in A_r(n-i)\}\right| = \sum_{i=1}^{r-1}|A_r(n-i)| = \sum_{i=1}^{r-1} f_{1,r}(n-i) \ . \qquad (3)$$

However, from the definition of $T_r(n)$ and (2) one sees that $f_{1,r}(n) = T_r(n-1)$, and so we deduce from (3) that

$$f_{2,r}(n) = \sum_{i=1}^{r-1} T_r(n-i-1) \ . \qquad (4)$$

Hence adding (2) and (4), we see that $T_r(n)$ satisfies the recurrence relation

$$T_r(n) = \sum_{i=1}^{r} T_r(n-i) \ ,$$

for $n > r$. Clearly $T_r(r) = 2^r - 1$ since the only binary string of length r having r runs of 1's is $\underbrace{11\ldots1}_{r1's}$, while $T_r(s) = 2^s$, for $s = 1, \ldots, r-1$ as all binary string of length less than r cannot contain r runs of 1's.

■

By applying Theorem 2.1 with $r = 2$, one can deduce again the well known result that there are F_{n+2} binary strings of length n having no consecutive 1's.

Corollary 2.1 *The number of binary strings of length $n \geq 2$ in which there are no pairs of consecutive ones is equal to F_{n+2}, where F_n is the n-th Fibonacci number.*

Proof: Setting $r = 2$ in Theorem 2.1 one finds $T_2(n)$ obeys the difference equation of the Fibonacci sequence as follows

$$T_2(n) = T_2(n-1) + T_2(n-2) \ ,$$

for $n > 2$, with initial conditions $T_2(1) = 2$ and $T_2(2) = 3$, which correspond to the shifted Fibonacci values F_3 and F_4 respectively. Thus the sequence $T_2(n)$ is identical to the shifted Fibonacci sequence F_{n+2}.

■

Before proving the main result of this section, consider the following example of a calculation of $T_r(n)$ using Theorem 2.1.

Example 2.1 *Suppose we wish to determine the number of binary strings of length say, $n = 6$, having no 3-runs of 1's, then from (1) we see $T_3(6)$ can be calculated via the difference equation*

$$T_3(n) = T_3(n-1) + T_3(n-2) + T_3(n-3) \ ,$$

for $n > 3$, with $T_3(1) = 2$, $T_3(2) = 4$ and $T_3(3) = 7$. A simple calculation yields there are $T_3(6) = T_3(5) + T_3(4) + T_3(3) = 24 + 13 + 7 = 44$ such binary strings, which we list in Table 1 for further study in Section 3.

0	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	1	1
0	0	0	1	0	0
0	0	0	1	1	0
0	0	0	1	0	1
0	0	1	0	0	0
0	0	1	0	1	0
0	0	1	0	0	1
0	0	1	0	1	1
0	0	1	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	0	0	1	0
0	1	0	0	0	1
0	1	0	0	1	1
0	1	0	1	0	0
0	1	0	1	1	0
0	1	0	1	0	1
0	1	1	0	0	0
0	1	1	0	1	0
0	1	1	0	0	1
0	1	1	0	1	1

1	0	0	0	0	0
1	0	0	0	1	0
1	0	0	0	0	1
1	0	0	0	1	1
1	0	0	1	0	0
1	0	0	1	1	0
1	0	0	1	0	1
1	0	1	0	0	0
1	0	1	0	1	0
1	0	1	0	0	1
1	0	1	0	1	1
1	0	1	1	0	0
1	0	1	1	0	1
1	1	0	0	0	0
1	1	0	0	1	0
1	1	0	0	0	1
1	1	0	0	1	1
1	1	0	1	0	0
1	1	0	1	1	0
1	1	0	1	0	1

Table 1: The 44 binary strings of length 6 having no 3-runs of 1's

Recall for a fixed integer $r \geq 2$, that an r-Fibonacci sequence $\{U_n\}$ is constructed from the difference equation

$$U_n = \sum_{i=1}^{r} U_{n-i} \ , \tag{5}$$

for $n \geq r$, with $U_0 = U_1 = \cdots = U_{r-2} = 0$ and $U_{r-1} = 1$, further-more a simple calculation yields that

$$U_{r+s} = 2^s \text{ for } s = 1, \ldots, r-1 \text{ and } U_{2r} = 2^r - 1. \tag{6}$$

Since for $r = 2$ we have that $U_n = F_n$, one can view Corollary 2.1 as a special case of a more general result we are about to prove, namely that the number of binary strings of length n having no r-runs of 1's, for $r \geq 3$, is equal to U_{n+r}.

Theorem 2.2 *For a fixed integer $r \geq 3$, the number of binary strings of length $n \geq r$ having no r-runs of 1's is equal to U_{n+r}, where $\{U_n\}$ is the r-Fibonacci sequence.*

Proof: Since for or a fixed integer $r \geq 3$, the r-Fibonacci sequence $\{U_n\}$ and the sequence $\{T_r(n)\}$ satisfy the same r-th order linear difference equation, it suffice to show that $T_r(n) = U_{n+r}$ for the r initial values of $n = 1, 2, \ldots, r$, however from (6) and the initial values of $T_r(n)$ stated in Theorem 2.1, we see this is the case. Thus the sequence $T_r(n)$ is identical to the shifted r-Fibonacci sequence U_{n+r}.

■

3 Characteristics of Binary Strings

Having shown that the number of binary strings without r-runs's of 1's can be expressed in terms of the r-Fibonacci sequence, we now turn our attention to a number of characteristics associated with these binary strings. These include the total number of zeros and the total number of runs, (i.e. maximal length substrings consisting either entirely of zeros or entirely of ones), contained in the U_{n+r} binary strings having no r-runs of 1's. Although such quantities were studied in [2] and expressed as functions of both the Fibonacci and Lucas numbers, we shall only determine difference equations for the calculation of these characteristics in the style of Theorem 2.1. To this end, it will first be necessary to establish the following technical lemma concerning the total number of zeros and runs contained in the set $S(n)$ of all binary strings of length n.

Lemma 3.1 *The total number of zeros and runs contained in the set $S(n)$ of all binary strings of length n, is given by $Z(n) = n2^{n-1}$ and $R(n) = (n+1)2^{n-1}$ respectively.*

Proof: Starting with the number of zeros, let $Z_1(n)$ and $Z_2(n)$ denote the number of zeros contained in the binary strings of $S(n)$, whose left hand entry contains a 0 and a 1 respectively. Now as $S(n) = \{0x : x \in S(n-1)\} \cup$

$\{1x : x \in S(n-1)\}$, for $n > 1$, observe each of the 2^{n-1} binary strings in $\{0x : x \in S(n-1)\}$ contributes an extra zero to the sum total of zeros already contained in $S(n-1)$, and so $Z_1(n) = 2^{n-1} + Z(n-1)$, while the number of zeros contained in the binary strings of $\{1x : x \in S(n-1)\}$ must be identical to the number in $S(n-1)$ and so $Z_2(n) = Z(n-1)$. Thus upon forming $Z(n) = Z_1(n) + Z_2(n)$ we see $Z(n) = 2Z(n-1) + 2^{n-1}$, but as $Z(1) = 1$ one finds after a standard calculation that $Z(n) = n2^{n-1}$, for $n \geq 1$. Next set $R_1(n)$ and $R_2(n)$ to be the number of runs contained in those binary strings of length n whose left hand entry contains a 0 and a 1 respectively. Now of the 2^{n-1} binary strings in $\{0x : x \in S(n-1)\}$, only half can contribute additional runs in the form of a single run of a zero, to the sum total of runs contained in $S(n-1)$ that is, the binary strings $0x$ for which x has a left hand entry of 1, and so $R_1(n) = R(n-1) + 2^{n-2}$. Similarly, only half of the binary strings in $\{1x : x \in S(n-1)\}$, can contribute additional runs in the form of a single run of a one, to the sum total of runs contained in $S(n-1)$ that is, the binary strings $1x$ for which x has a left hand entry of 0, and so $R_2(n) = R(n-1) + 2^{n-2}$. Thus upon forming $R(n) = R_1(n) + R_2(n)$ we see $R(n) = 2R(n-1) + 2^{n-1}$, but as $R(1) = 2$ one finds after a standard calculation that $R(n) = (n+1)2^{n-1}$, for $n \geq 1$.

■

We now examine the total number of zeros, denoted $Z_r(n)$, contained in the U_{n+r} binary strings having no r-runs of ones.

Theorem 3.1 *For a fixed integer $r \geq 3$, the total number of zeros that occur in the U_{n+r} binary strings of length n having no r-runs of 1's, satisfies the following r-th order linear difference equation*

$$Z_r(n) = \sum_{i=1}^{r} Z_r(n-i) + U_{n+r} \ , \tag{7}$$

for $n > r$, with the r initial conditions $Z_r(s) = s2^{s-1}$, for $s = 1, 2, \ldots, r$.

Proof: We first construct the difference equation for $Z_r(n)$, where $n > r$. Let $Z_{1,r}(n)$ and $Z_{2,r}(n)$ denote the total number of zeros that occur in those binary strings contained in the sets $A_r(n)$ and $B_r(n)$ respectively. Considering an arbitrary binary string $x \in A_r(n)$, recall that x can be uniquely written as the concatenation $0y$, where $y \in S_r(n-1) = A_r(n-1) \cup B_r(n-1)$. Thus each $x \in A_r(n)$ contributes an extra zero to the sum total of zeros already contained in the substrings of $S_r(n-1)$, and so recalling from (2) and Theorem 2.2 that $U_{n-1+r} = T_r(n-1) = f_{1,r}(n)$, observe

$$\begin{aligned} Z_{1,r}(n) = Z_r(n-1) + |A_r(n)| &= Z_r(n-1) + f_{1,r}(n) \\ &= Z_r(n-1) + T_r(n-1) \\ &= Z_r(n-1) + U_{n-1+r} \ . \end{aligned} \tag{8}$$

Alternatively, if $x \in B_r(n) = \bigcup_{i=1}^{r-1}\{\underbrace{11\cdots1}_{i1's} y : y \in A_r(n-i)\}$ then again as every $y \in A_r(n-i)$ can be uniquely written as the concatenation $0y'$, where $y' \in S_r(n-i-1) = A_r(n-i-1) \cup B_r(n-i-1)$ observe the number of zeros in $\{\underbrace{11\cdots1}_{i1's} y : y \in A_r(n-i)\}$ is equal to the sum total of zeros contained in the substrings of $S_r(n-i-1)$, together with each additional zero contributed from the first zero of $y = 0y'$ that is, $Z_r(n-i-1) + |A_r(n-i)|$, and so again recalling from (2) and Theorem 2.2 that $U_{n-i-1+r} = T_r(n-i-1) = f_{1,r}(n-i)$, observe

$$\begin{aligned} Z_{2,r}(n) = \sum_{i=1}^{r-1}(Z_r(n-i-1) + |A_r(n-i)|) &= \sum_{i=1}^{r-1}(Z_r(n-i-1) + f_{1,r}(n-i)) \\ &= \sum_{i=1}^{r-1}(Z_r(n-i-1) + T_r(n-i-1)) \\ &= \sum_{i=1}^{r-1}(Z_r(n-i-1) + U_{n-i-1+r}) \ . \end{aligned} \tag{9}$$

Hence adding (8) and (9) and applying the difference equation of (5), we see $Z_r(n)$ satisfies the difference equation

$$\begin{aligned} Z_r(n) = \sum_{i=1}^{r}(Z_r(n-i) + U_{n-i+r}) &= \sum_{i=1}^{r} Z_r(n-i) + \sum_{i=1}^{r} U_{n-i+r} \\ &= \sum_{i=1}^{r} Z_r(n-i) + U_{n+r} \ , \end{aligned}$$

for $n > r$. Clearly the number of zeros that occur in the binary strings of length less than or equal to r, and not having r-runs of 1's, must be equal to the number of zeros that occur in all binary strings having corresponding lengths less than or equal to r. Thus by Lemma 3.1 $Z_r(s) = Z(s) = s2^{s-1}$, for $s = 1, \ldots, r$.

■

Before examining the next characteristic, consider the following example of a calculation of $Z_r(n)$ using Theorem 3.1.

Example 3.1 *Returning to the binary strings of Example 2.1, suppose we wish to determine the total number of zeros present in Table 1, then from (7) we see* $Z_3(6)$ *can be calculated via the difference equation*

$$Z_3(n) = Z_3(n-1) + Z_3(n-2) + Z_3(n-3) + U_{n+3} \ ,$$

for $n > 3$*, with* $Z_3(1) = 1$*,* $Z_3(2) = 4$ *and* $Z_3(3) = 12$*. A simple calculation yields there are* $Z_3(6) = Z_3(5) + Z_3(4) + Z_3(3) + U_9 = 70 + 30 + 12 + 44 = 156$ *zeros contained in Table 1, which can be verified with a manual count.*

Finally, in like manner we examine the total number of runs, denoted $R_r(n)$, contained in the U_{n+r} binary strings having no r-runs of 1's.

Theorem 3.2 *For a fixed integer* $r \geq 3$*, the total number of runs that occur in the* U_{n+r} *binary strings of length n having no r-runs of* 1*'s, satisfies the following r-th order linear difference equation*

$$R_r(n) = \sum_{i=1}^{r} R_r(n-i) + U_{n+r} - U_{n-1} \ , \tag{10}$$

for $n > r$*, with the r initial conditions* $R_r(s) = (s+1)2^{s-1}$*, for* $s = 1, 2, \ldots, r-1$*, and* $R_r(r) = (r+1)2^{r-1} - 1$*.*

Proof: We first construct the difference equation for $R_r(n)$, where $n > r$. Let $R_{1,r}(n)$ and $R_{2,r}(n)$ denote the total number of runs that occur in those binary strings contained in the sets $A_r(n)$ and $B_r(n)$ respectively. Considering an arbitrary binary string $x \in A_r(n)$, recall that x can be uniquely written as $0y$, where $y \in S_r(n-1) = A_r(n-1) \cup B_r(n-1)$. Now the binary strings $x = 0y$, where $y \in A_r(n-1)$ contribute no additional runs to those in $A_r(n-1)$, since the left hand entry of y is a zero, however each binary string $x = 0y$, where $y \in B_r(n-1)$ will contribute an additional run of a single run of a zero to those in $B_r(n-1)$, since the left hand entry of y is a one, and so

$$\begin{aligned} R_{1,r}(n) &= R_{1,r}(n-1) + R_{2,r}(n-1) + |B_r(n-1)| \\ &= R_r(n-1) + f_{2,r}(n-1) \ . \end{aligned} \tag{11}$$

Alternatively, if $x \in B_r(n) = \bigcup_{i=1}^{r-1} \{\underbrace{11\cdots1}_{i1's} y : y \in A_r(n-i)\}$ then again as every $y \in A_r(n-i)$ can be uniquely written as $0y'$, where $y' \in S_r(n-i-1) =$

$A_r(n-i-1) \cup B_r(n-i-1)$, observe each of the $r-1$ sets $\{\underbrace{11\cdots1}_{i1's}y : y \in A_r(n-i)\}$ are of the form

$$\{\underbrace{11\cdots1}_{i1's}0y' : y' \in A_r(n-i-1)\} \cup \{\underbrace{11\cdots1}_{i1's}0y' : y' \in B_r(n-i-1)\} \ . \quad (12)$$

Now the binary strings $\underbrace{11\cdots1}_{i1's}0y'$, where $y' \in A_r(n-i-1)$, will each contribute one additional run of i ones to the sum total contained in $A_r(n-i-1)$, since the left hand entry of y' is a zero, and so the number of runs in $\{\underbrace{11\cdots1}_{i1's}0y' : y' \in A_r(n-i-1)\}$ is $R_{1,r}(n-i-1)+|A_r(n-i-1)| = R_{1,r}(n-i-1)+f_{1,r}(n-i-1)$. However the binary strings $\underbrace{11\cdots1}_{i1's}0y'$, where $y' \in B_r(n-i-1)$, will each contribute two additional runs in the form of one run of i ones, and one single run of a zero to the sum total in $B_r(n-i-1)$, since the left hand entry of y' is a one, and so the number of runs in $\{\underbrace{11\cdots1}_{i1's}0y' : y' \in B_r(n-i-1)\}$ is $R_{2,r}(n-i-1)+2|B_r(n-i-1)| = R_{2,r}(n-i-1)+2f_{2,r}(n-i-1)$. Consequently, by recalling from (2) that $T_r(n-i-1) = f_{1,r}(n-i) = f_{1,r}(n-i-1)+f_{2,r}(n-i-1)$, we see from (12) that the total number of runs in $B_r(n)$ is

$$\begin{aligned} R_{2,r}(n) &= \sum_{i=1}^{r-1} R_{1,r}(n-i-1) + R_{2,r}(n-i-1) + f_{1,r}(n-i-1) + 2f_{2,r}(n-i-1) \\ &= \sum_{i=1}^{r-1} R_r(n-i-1) + f_{1,r}(n-i) + f_{2,r}(n-i-1) \\ &= \sum_{i=1}^{r-1} R_r(n-i-1) + T_r(n-i-1) + f_{2,r}(n-i-1) \ . \end{aligned} \quad (13)$$

Adding (11) and (13), we deduce $R_r(n)$ satisfies the difference equation

$$R_r(n) = \sum_{i=1}^{r} R_r(n-i) + \sum_{i=1}^{r-1} T_r(n-i-1) + \sum_{i=1}^{r} f_{2,r}(n-i) \ , \quad (14)$$

for $n > r$. Our task is thus reduced to showing that the sum of the last two summations in (14) evaluates to $U_{n+r} - U_{n-1}$. To this end, as $n > r$ we see from the difference equation in (1) that

$$\sum_{i=1}^{r-1} T_r(n-i-1) = T_r(n) - T_r(n-1) \ ,$$

while from (2), as $f_{1,r}(n-i+1) = f_{1,r}(n-i) + f_{2,r}(n-i)$, and as $T_r(m-1) = f_{1,r}(m)$, observe

$$\begin{aligned}\sum_{i=1}^{r} f_{2,r}(n-i) &= \sum_{i=1}^{r}(f_{1,r}(n-i+1) - f_{1,r}(n-i)) \\ &= f_{1,r}(n) - f_{1,r}(n-r) \\ &= T_r(n-1) - T_r(n-r-1) \ .\end{aligned}$$

Consequently from Theorem 2.2 as $T_r(n) = U_{n+r}$, the last two summations in (14) reduce to $T_r(n) - T_r(n-r-1) = U_{n+r} - U_{n-1}$, as required. Clearly the number of runs that occur in the binary strings of length strictly less than r, and not having r-runs of 1's, must be equal to the number of runs that occur in all binary strings having corresponding lengths strictly less than r. Thus by Lemma 3.1, $R_r(s) = R(s) = (s+1)2^{s-1}$, for $s = 1, \ldots, r-1$. While as the only binary string of length r contributing a single run of r 1's is $\underbrace{11\cdots1}_{r1's}$, the total number of runs in the remaining $T_r(r) = 2^r - 1$ binary strings must be $R_r(r) = R(r) - 1 = (r+1)2^{r-1} - 1$.

■

To conclude this section, consider the following example of a calculation of $R_r(n)$ using Theorem 3.2.

Example 3.2 *Again returning to the binary strings of Example 2.1, suppose we wish to determine the total number of runs present in Table 1, then from (10) we see* $R_3(6)$ *can be calculated via the difference equation*

$$R_3(n) = R_3(n-1) + R_3(n-2) + R_3(n-3) + U_{n+3} - U_{n-1} \ ,$$

for $n > 3$, *with* $R_3(1) = 2$, $R_3(2) = 6$ *and* $R_3(3) = 15$. *A simple calculation yields there are* $R_3(6) = R_3(5)+R_3(4)+R_3(3)+U_9-U_5 = 78+35+15+44-4 = 168$, *runs contained in Table 1, which can be verified with a manual count.*

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References

[1] R. P. Grimaldi, *Discrete and Combinatorial Mathematics*. 2nd ed., Addison-Wesley, (1989).

[2] R. P. Grimaldi, S. Heubach, Binary Strings Without Odd Runs of Zeros. *Ars Combinatoria*, Vol. 75, pp. 241-255, (2005).

[3] R. P. Grimaldi, Ternary Strings With No Consecutive 1's. *Ars Combinatoria*, Vol. 89, pp. 321-343, (2008).

[4] B. Hayes, D. S. Pearson, M. Andreoli, Solutions 10196. *The American Mathematical Monthly*, Vol. 101, No. 4, pp. 363-365, (1994).

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