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On Best Proximity Point Theorems for New Cyclic Maps

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Abstract

In this paper, we first introduce the concept of $\mathcal{MT}-K$ condition. Some best proximity point theorems for mappings satisfying $\mathcal{MT}-K$ condition instead of K-cyclic mappings are established in metric spaces. Our results generalize and improve some main results in [5] and references therein.

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let A and B be nonempty subsets of a nonempty set E. A map $S : A\cup B \to A\cup B$ is called a *cyclic* map if $S(A) \subset B$ and $S(B) \subset A$. Let (X, d) be a metric space and $T : A \cup B \to A \cup B$ a cyclic map. For any nonempty subsets A and B of X, let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A \cup B$ is called to be a best proximity point for T if $d(x, Tx) = dist(A, B)$.

Definition 1.1. **[12]** Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a cyclic contraction if the following conditions hold:

(1) $T(A) \subset B$ and $T(B) \subset A$;

(2) There exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (1-k)dist(A, B)$ for all $x \in A, y \in B$.

Remark 1.1. Let A and B be nonempty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic contraction. If $A \cap B \neq \emptyset$, then $dist(A, B) = 0$ and T is a contraction on the complete metric space $(A \cap B, d)$. Hence, applying the Banach contraction principle, we know that T has a unique fixed point in $A \cap B$.

Since the equation $Tx = x$, where T is a self-mapping, does not necessarily have a solution, we can turn to the existence of approximate solutions for the best proximity point. Recently, the existence, uniqueness and convergence of iterates to the best proximity point were investigated by many authors; see [1-5,12-33] and references therein. In [12], Eldred and Veeramani first proved the following interesting best proximity point theorem.

Theorem EV. **[12, Proposition 3.2]** Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.

In this paper, we first introduce the concept of $\mathcal{MT} - K$ condition. Some best proximity point theorems for mappings satisfying $\mathcal{MT} - K$ condition (see Def. 2.4 below) instead of K-cyclic mappings are established in metric spaces. Our results generalize and improve some main results in [5] and references therein.

2. Preliminaries

For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0 < |x-c| < \varepsilon} f(x)$$

and

$$\limsup_{x\to c^+} f(x) = \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x).$$

Definition 1.1. [6-7, 15-17] A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-*function* if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$).

It is obvious that if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example 1.1. [11, 16, 17] Let $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t} & \text{, if } t \in (0, \frac{\pi}{2}] \\ 0 & \text{, otherwise.} \end{cases}$$

Since $\limsup_{s\to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

Very recently, Du [11] first proved some characterizations of $\mathcal{MT}$-functions.

Theorem D. [11, Theorem 2.1] Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.

(a) φ is an $\mathcal{MT}$-function.

(b) For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.

(c) For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.

(d) For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.

(e) For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.

(f) For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

(g) φ is a **function of contractive factor [8]**; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

For a cyclic map $T : A \cup B \to A \cup B$, Du et al. [15] introduced the notion of $\mathcal{MT}$-cyclic contraction with respect to φ on $A \cup B$ and then they proved new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}$-cyclic contractions.

Definition 2.1. **[15]** Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MT1) $T(A) \subset B$ and $T(B) \subset A$;

(MT2) there exists a $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0, 1)$ such that

$$d(Tx, Ty) \le \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))dist(A, B) \quad \text{for any } x \in A \text{ and } y \in B,$$

then T is called a $\mathcal{MT}$-cyclic contraction with respect to φ on $A \cup B$.

Afterward, Lakzian and Lin in [16] using of concept of weak $\mathcal{MT}$-cyclic Kannan contractions with respect to φ on $A \cup B$ established some new convergent and existence theorems of best proximity point theorems for these contractions in uniformly Banach spaces that generalized theorem by Petric [18].

Definition 2.1.[16] Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MTK1) $T(A) \subset B$ and $T(B) \subset A$;

(MTK2) there exists a $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0, 1)$ such that

$$d(Tx, Ty) \le \frac{1}{2}\varphi(d(x, y))[d(x, Tx) + d(y, Ty)] + (1 - \varphi(d(x, y)))dist(A, B)$$

$$\text{for any } x \in A \text{ and } y \in B,$$

then T is called a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to φ on $A \cup B$.

In [5], the authors, independently, for two mappings S and T that $T : A \to B$ and $T : B \to A$, this notion is defined by S. Sadiq Basha et al.

Definition 2.2.[5] A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a K-Cyclic mapping between A and B if there exists a nonnegative real number $k < 1/2$ such that

$$d(Tx, Sy) \leq k[d(x, Tx) + d(y, Sy)] + (1 - 2k)d(A, B),$$

for all $x \in A$ and $y \in B$.

Motivated by the concepts of K-cyclic mappings and $\mathcal{MT}$-function, we first introduce the concept of $\mathcal{MT} - K$ condition as follows:

Definition 2.3. Let A and B be non-empty subsets of a metric space (X, d) and $T : A \to B$ and $S : B \to A$ be maps. We call the pair of maps T and S satisfy $\mathcal{MT} - K$ condition if there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Sy) \leq \frac{1}{2}\varphi(d(x, y))[d(x, Tx) + d(y, Sy)] + (1 - \varphi(d(x, y)))d(A, B),$$

for all $x \in A$, $y \in B$.

3. Main results

In this section, we first prove an existence theorem.

Theorem 3.1. Let A and B be nonempty subsets of a metric space (X, d). Let $T : A \to B$ and $S : B \to A$ be maps. If the pair of maps T and S satisfy $\mathcal{MT} - K$ condition, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Proof. Since maps T and S satisfy $\mathcal{MT} - K$ condition, there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Sy) \leq \frac{1}{2}\varphi(d(x, y))[d(x, Tx) + d(y, Sy)] + (1 - \varphi(d(x, y)))d(A, B), \quad (1)$$

for all $x \in A$, $y \in B$. Let $x_0 \in A$ be given. Define $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$ for each $n \in \mathbb{N} \cup \{\nvdash\}$. Then $x_{2n} \in A$ and $x_{2n+1} \in B$ for each $n \in \mathbb{N} \cup \{\nvdash\}$. By (1), we have

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Sx_1) \\ &\leq \frac{1}{2}\varphi(d(x_0, x_1))[d(x_0, Tx_0) + d(x_1, Sx_1)] + (1 - \varphi(d(x_0, x_1)))d(A, B) \\ &= \frac{1}{2}\varphi(d(x_0, x_1))[d(x_0, x_1) + d(x_1, x_2)] + (1 - \varphi(d(x_0, x_1)))d(A, B). \end{aligned}$$

It follows that $[1-\frac{1}{2}\varphi(d(x_0,x_1))]d(x_1,x_2) \leq [\frac{1}{2}\varphi(d(x_0,x_1))]d(x_0,x_1)+(1-\varphi(d(x_0,x_1)))d(A,B)$,

which implies that

$$d(x_1,x_2) \leq \frac{\varphi(d(x_0,x_1))}{2-\varphi(d(x_0,x_1))}d(x_0,x_1)+[1-\frac{\varphi(d(x_0,x_1))}{2-\varphi(d(x_0,x_1))}]d(A,B). \quad (2)$$

By (2), we obtain that

$$d(x_1,x_2)-d(A,B) \leq \frac{\varphi(d(x_0,x_1))}{2-\varphi(d(x_0,x_1))}(d(x_0,x_1)-d(A,B)).$$

By (1) again, we have

$$\begin{aligned} d(x_2,x_3) &= d(Sx_1,Tx_2)=d(Tx_2,Sx_1) \\ &\leq \frac{1}{2}\varphi(d(x_2,x_1))[d(x_2,Tx_2)+d(x_1,Sx_1)]+[1-\varphi(d(x_2,x_1))]d(A,B) \\ &= \frac{1}{2}\varphi(d(x_2,x_1))[d(x_2,x_3)+d(x_1,x_2)]+[1-\varphi(d(x_2,x_1))]d(A,B), \end{aligned}$$

which implies

$$[1-\frac{1}{2}\varphi(d(x_2,x_1))]d(x_2,x_3) \leq \frac{1}{2}\varphi(d(x_2,x_1))d(x_1,x_2)+[1-\varphi(d(x_2,x_1))]d(A,B),$$

and hence

$$d(x_2,x_3) \leq \frac{(d(x_2,x_1))}{2-\varphi(d(x_2,x_1))}d(x_2,x_1)+[1-\frac{\varphi(d(x_2,x_1))}{2-\varphi(d(x_2,x_1))}]d(A,B)$$

or

$$d(x_3,x_2)-d(A,B) \leq \frac{\varphi(d(x_2,x_1))}{2-\varphi(d(x_2,x_1))}(d(x_2,x_1)-d(A,B)).$$

By induction, we have

$$d(x_n,x_{n+1})-d(A,B) \leq \frac{\varphi(d(x_{n-1},x_n))}{2-\varphi(d(x_{n-1},x_n))}(d(x_{n-1},x_n)-d(A,B)). \quad (3)$$

Since $\varphi(t)<1$ for all $t\in[0,\infty)$, we have $\frac{\varphi(t)}{2-\varphi(t)}<1$ for all $t\in[0,\infty)$. By (3), we get

$$d(x_n, x_{n+1}) - d(A,B) < d(x_{n-1}, x_n) - d(A,B),$$

which implies $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. So $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence in $[0,\infty)$. Since φ is an $\mathcal{MT}$-function, by Theorem D, we obtain

$$0 \le \sup_{n\in\mathbb{N}} \varphi(d(x_n, x_{n+1})) < 1.$$

Let $\lambda = \sup_{n\in\mathbb{N}} \varphi(d(x_n, x_{n+1}))$. So $0 \le \lambda < 1$. Since $\varphi(d(x_n, x_{n+1})) \le \lambda$, we have

$$2 - \varphi(d(x_n, x_{n+1})) \ge 2 - \lambda.$$

Then

$$\frac{\varphi(d(x_n, x_{n+1}))}{2 - \varphi(d(x_n, x_{n+1}))} \le \frac{\lambda}{2-\lambda},$$

for all $n \in \mathbb{N}$.

So

$$0 \le \sup_{n\in\mathbb{N}} \frac{\varphi(d(x_n, x_{n+1}))}{2 - \varphi(d(x_n, x_{n+1}))} \le \frac{\lambda}{2-\lambda} < 1.$$

Let

$$\gamma = \sup_{n\in\mathbb{N}} \frac{\varphi(d(x_n, x_{n+1}))}{2 - \varphi(d(x_n, x_{n+1}))}.$$

Then $\gamma \in [0,1)$. By (3) again, it follows that

$$\begin{aligned}
d(x_n, x_{n+1}) - d(A,B) &\le \frac{\varphi(d(x_(n-1), x_n))}{2 - \varphi(d(x_{n-1}, x_n))}(d(x_{n-1}, x_n) - d(A,B)) \\
&\le \gamma(d(x_{n-1}, x_n) - d(A,B)) \\
&\le \gamma^2(d(x_{n-2}, x_{n-1}) - d(A,B)) \\
&\le \cdots \\
&\le \gamma^n(d(x_0, x_1) - d(A,B)).
\end{aligned}$$

Since $\gamma \in [0,1)$, we have $\lim_{n\to\infty} \gamma^n = 0$. By (4),

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

The proof is completed. □

As a direct consequence of Theorem 3.1, we obtain the following.

Corollary 3.1. [**5**] Let A and B be nonempty subsets of a metric space (X, d). Let $T : A \to B$ and $S : B \to A$ be maps. If the pair of maps T and S satisfy K condition, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Theorem 3.2. Let A and B be two non-empty subsets of a metric space (X, d) and $T : A \to B$ and $S : B \to A$ be maps. If the pair of maps T and S satisfies $\mathcal{MT} - K$ condition. If $x_0 \in A$, define $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$, $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is bounded.

Proof. By Theorem 3.1, we have

$$\lim_{n\to} d(x_n, x_{n+1}) = d(A, B).$$

Since $\{d(x_{2n-1}, x_{2n})\}$ is a subsequence of $\{d(x_n, x_n + 1)\}$, we have

$$\lim_{n\to} d(x_{2n-1}, x_{2n}) = d(A, B).$$

Hence $\{d(x_{2n-1}, x_{2n})\}$ is bounded. So there exists $L > 0$ such that

$$d(x_{2n-1}, x_{2n}) \le L,$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{2n}, Tx_0) &= d(Sx_{2n-1}, Tx_0) \\ &\le \frac{1}{2}\varphi(d(x_{2n-1}, x_0))[d(x_{2n-1}, Sx_{2n-1}) + d(x_0, Tx_0)] + [1 - \varphi(d(x_{2n-1}, x_0))]d(A, B) \\ &< \frac{1}{2}[d(x_{2n-1}, x_{2n}) + d(x_0, Tx_0)] + d(A, B) \\ &\le \frac{1}{2}[L + d(x_0, Tx_0)] + d(A, B). \end{aligned}$$

Let

$$M = \frac{1}{2}[L + d(x_0, Tx_0)] + d(A, B).$$

Hence $x_{2n} \in \overline{B}(Tx_0, M)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, since

$$d(x_{2n+1}, Tx_0) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_0) \leq L + M.$$

We obtain $x_{2n+1} \in \overline{B}(Tx_0, L+M)$ for all $n \in \mathbb{N}$. On the other hand, since

$$x_{2n} \in \overline{B}(Tx_0, M) \subseteq \overline{B}(Tx_0, L+M),$$

for all $n \in \mathbb{N}$, we also have $x_{2n} \in \overline{B}(Tx_0, L+M)$ for all $n \in \mathbb{N}$. Hence

$$x_n \in \overline{B}(Tx_0, L+M)$$

for all $n \in \mathbb{N}$, which means that $\{x_n\}$ is bounded. The proof is completed. □

As a direct consequence of Theorem 3.2, we obtain the following.

Corollary 3.2. **[5]** Let A and B be two non-empty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form a K-Cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then the sequence $\{x_n\}$ is bounded.

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