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Neighborhood Connected 2-Domination in Graphs

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Abstract

Let $G = (V, E)$ be a connected graph. A set $S \subseteq V$ is called the neighborhood connected 2-dominating set (nc2d-set) of a graph G if every vertex in $V - S$ is adjacent to at least two vertices in S and the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a nc2d-set of G is called the neighborhood connected 2-domination number of G and is denoted by $\gamma_{2nc}(G)$. In this paper we initiate a study of this parameter.

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1 Introduction

The graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2] and Haynes et.al [4, 5].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N(S) = \bigcup\limits_{v \in S} d(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $\text{pn}[u, S] = \{v : N[v] \cap S = \{u\}\}$. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the graph H by pasting m_i edges to the vertex $v_i \in V(H)$, $1 \leq i \leq n$.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al., Fink and Jacobson [see 4] introduced the concept of k-domination in graphs. A dominating set S of G is called a k- dominating set if every vertex in $V - S$ is adjacent to at least k vertices in S. The minimum cardinality of a k-dominating set is called k-domination number of G and is denoted by $\gamma_k(G)$. F. Harary and T.W. Haynes [3] introduced the concept of double domination in graphs. A dominating set S of G is called a double dominating set if every vertex in $V - S$ is adjacent to at least two vertices in S and every vertex in S is adjacent to at least one vertex in S. The minimum cardinality of a double dominating set is called double domination number of G and is denoted by $dd(G)$. S. Arumugam and C. Sivagnanam [1] introduced the concept of neighborhood connected domination in graphs. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. In this paper we introduce the concept of neighborhood connected 2-domination and initiate a study of the corresponding parameter. We need the following theorems.

Theorem 1.1. [1] *For a path* P_n, $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$.

Theorem 1.2. [1] *For the cycle* C_n on n vertices

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 (\text{mod } 4) \end{cases}$$

2 Main Results

Definition 2.1. A set $S \subseteq V$ is called the neighborhood connected 2-dominating set (nc2d-set) of a graph G if every vertex in $V - S$ is adjacent to at least two vertices in S and the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a nc2d-set of G is called the neighborhood connected 2-domination number of G and is denoted by $\gamma_{2nc}(G)$.

Remark 2.2. (i) Clearly $\gamma_{2nc}(G) \geq \gamma_{nc}(G) \geq \gamma(G)$ and $\gamma_{2nc}(G) \geq \gamma_2(G)$.
(ii) A graph G has $\gamma_{2nc}(G) = 2$ if and only if there exist two vertices $u, v \in V$ such that $(a)\deg u = \deg v = n - 1$ or $(b)\deg u = \deg v = n - 2$, $uv \notin E(G)$ with $\{u, v\}$ is not a vertex cut of G. Thus $\gamma_{2nc}(G) = 2$ if and only if G is isomorphic to either $H + K_2$ for some graph H or $H + \overline{K_2}$ for some connected graph H.

Examples:
(1) $\gamma_{2nc}(K_n) = 2$
(2) $\gamma_{2nc}(K_{1,n-1}) = n - 1, \quad n \geq 3$
(3) Let $K_{r,s}$ be a complete bipartite graph and not a star then

$$\gamma_{2nc}(K_{r,s}) = \begin{cases} 3 & \text{r or s} = 2 \\ 4 & \text{r,s } \geq 3 \end{cases}$$

(4) $\gamma_{2nc}(W_n) = \lceil \frac{n}{3} \rceil + 1$

Theorem 2.3. For any non trivial path P_n, $\gamma_{2nc}(P_n) = \lceil \frac{2n}{3} \rceil$.

Proof. Let $P_n = (v_1, v_2, \cdots, v_n)$ and $n = 3k + r$ where $0 \leq r \leq 2$. Let $S = \{v_i \in V : i = 3j, 3j + 1, \ 0 \leq j \leq k\}$

$$\text{Let} \quad S_1 = \begin{cases} S & \text{if} \quad n \equiv 0, 1 \ (\text{mod } 3) \\ S \cup \{v_n\} & \text{if} \quad n \equiv 2 \quad (\text{mod } 3) \end{cases}$$

Clearly S_1 is a nc2d-set of P_n and hence $\gamma_{2nc}(P_n) \leq \lceil \frac{2n}{3} \rceil$. Further if S is any γ_{2nc}-set of P_n, then $N(S)$ contains all the internal vertices of P_n and hence $|S| \geq \lceil \frac{2n}{3} \rceil$. Thus $\gamma_{2nc}(P_n) = \lceil \frac{2n}{3} \rceil$. □

Corollary 2.4. For any non trivial path P_n, $\gamma_{2nc}(P_n) = \gamma_{nc}(P_n)$ if and only if $n = 3$.

Proof. Since $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$ the corollary follows. □

Theorem 2.5. For the cycle C_n on n vertices

$$\gamma_{2nc}(C_n) = \begin{cases} \lfloor \frac{2n}{3} \rfloor & \text{if} \quad n \equiv 2 \pmod 3 \\ \\ \lceil \frac{2n}{3} \rceil & \text{if} \quad n \not\equiv 2 \pmod 3 \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \cdots, v_n, v_1)$ and $n = 3k + r$, where $0 \leq r \leq 2$. Let $S = \{v_i : i = 3j+1,\ 3j+2,\ \ 0 \leq j \leq\ \ k-1\}$.

$$\text{Let} \quad S_1 = \begin{cases} S & \text{if} \quad n \equiv 0 (\text{mod } 3) \\ \\ S \cup \{v_n\} & \text{if} \quad n \equiv 1 (\text{mod } 3) \\ \\ S \cup \{v_{n-1}\} & \text{if} \quad n \equiv 2 (\text{mod } 3) \end{cases}$$

Clearly S_1 is a nc2d-set of C_n and hence

$$\gamma_{2nc}(C_n) \leq \begin{cases} \lfloor \frac{2n}{3} \rfloor & \text{if} \quad n \equiv 2\ (\text{mod } 3) \\ \\ \lceil \frac{2n}{3} \rceil & \text{if} \quad n \not\equiv 2\ (\text{mod } 3). \end{cases}$$

Now, let S be any γ_{2nc}-set of C_n. Then $\langle S \rangle$ contains at most one isolated vertex and

$$N(S) = \begin{cases} P_{n-1} \text{ if } n \equiv 2\ (\text{mod } 3) \\ \\ C_n \text{ if } n \not\equiv 2 (\text{mod } 3) \end{cases}$$

Hence

$$|S| \geq \begin{cases} \lfloor \frac{2n}{3} \rfloor & \text{if} \quad n \equiv 2\ (\text{mod } 3) \\ \\ \lceil \frac{2n}{3} \rceil & \text{if} \quad n \not\equiv 2\ (\text{mod } 3) \end{cases}$$

and the result follows. □

Corollary 2.6. $\gamma_{2nc}(C_n) = \gamma_{nc}(C_n)$ if and only if $n = 5$.

Proof. Since

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 (\text{mod } 4) \end{cases}$$

the result follows. □

We now proceed to obtain a characterization of minimal nc2d-sets.

Lemma 2.7. A superset of a nc2d-set is a nc2d-set.

Proof. Let S be a nc2d-set of a graph G and let $S_1 = S \cup \{v\}$, where $v \in V - S$. Clearly $v \in N(S)$ and S_1 is a 2-dominating set of G. Now, let $x, y \in N(S_1)$. If $x, y \in N(S)$ then any $x - y$ path in $N(S)$ is a $x - y$ path in $N(S_1)$. If $x \in N(S)$ and $y \notin N(S)$ then $y \in N(v)$ and any $x - v$ path in $N(S)$ followed by the edge vy is a $x - y$ path in $N(S_1)$. Also if $x, y \notin N(S)$ then (x, v, y) is a $x - y$ path in $N(S_1)$. Thus $\langle N(S_1) \rangle$ is connected so that S_1 is a nc2d-set of G. □

Theorem 2.8. A nc2d-set S of a graph G is a minimal nc2d-set if and only if for every $u \in S$, one of the following holds.

(i) $|N(u) \cap S| \leq 1$

(ii) there exists a vertex $v \in (V - S) \cap N(u)$ such that $|N(v) \cap S| = 2$

(iii) there exist two vertices $x, y \in N(S)$ such that every $x - y$ path in $\langle N(S) \rangle$ contains at least one vertex of $N(S) - N(S - \{u\})$.

Proof. Let S be a minimal nc2d-set and let $u \in S$. Let $S_1 = S - \{u\}$. Then S_1 is not a nc2d-set. This gives either S_1 is not a 2-dominating set or $\langle N(S) \rangle$ is disconnected. If S_1 is not a 2-dominating set then there exists a vertex $v \in V - S_1$ such that $|N(v) \cap S_1| \leq 1$. If $v = u$ then $|N(u) \cap (S - \{u\})| \leq 1$ which gives $|N(u) \cap S| \leq 1$. Suppose $v \neq u$. If $|N(v) \cap S_1| < 1$ then $|N(v) \cap S| \leq 1$ and hence S is not a 2-dominating set which is a contradiction. Hence $|N(v) \cap S_1| = 1$. Thus $v \in N(u)$. So $v \in (V - S) \cap N(u)$ such that $|N(v) \cap S| = 2$. If $\langle N(S_1) \rangle$ is disconnected then there exist two vertices $x, y \in N(S_1)$ such that there is no $x - y$ path in $\langle N(S_1) \rangle$. Since $\langle N(S) \rangle$ is connected, it follows that every $x - y$ path in $\langle N(S) \rangle$ contains at least one vertex of $N(S) - N(S - \{u\})$. Conversely, if S is a nc2d-set of G satisfying the conditions of the theorem, then S is 1-minimal and hence the result follows from lemma 2.7. □

Remark 2.9. Any nc2d-set contains all the pendant vertices of the graph.

Remark 2.10. For any graph G, $\gamma_{2nc}(G) \leq n$ and equality holds if and only if G is isomorphic to K_2.

Theorem 2.11. Let T be a tree with $n \geq 3$ vertices. Then $\gamma_{2nc}(T) = n-1$ if and only if T is a star or a bistar $B(n-3,1)$ or a tree obtained from a bistar by subdividing the edge of maximum degree once.

Proof. Let u be a support with maximum degree. Suppose there exists a vertex $v \in V(T)$ such that $d(u,v) \geq 4$. Let $(u, x_1, x_2, \cdots, x_k, v)$, $k \geq 3$ be the shortest $u-v$ path then $S_1 = V - \{u, x_k\}$ is a nc2d-set of T which is a contradiction. Hence $d(u,v) \leq 3$ for all $v \in V(T)$.

Case 1. $d(u,v) = 3$ for some $v \in V(T)$

Suppose there exists an vertex $w \in V(T)$, $w \neq v$ such that $d(u,v) = d(u,w) = 3$. Let P_1 be the $u-v$ path and P_2 be the $u-w$ path. Let $P_1 = (u, v_1, v_2, v)$ and $P_2 = (u, w_1, w_2, w)$. If $v_1 \neq w_1$ then $V - \{v_1, w_1\}$ is a nc2d-set of T which is a contradiction. If $v_1 = w_1$ and $v_2 \neq w_2$ then $V - \{v_2, w_2\}$ is a nc2d-set of T which is a contradiction. Hence all the pendant vertices w such that $d(u,w) = 3$ are adjacent to the same support. Let it be x. Let $P = (u, v_1, x)$ be the unique $u-x$ path in T. Let $y \in N(u) - \{v_1\}$. If $\deg y \geq 2$ then $V - \{x, y\}$ is a nc2d-set of T which is a contradiction. Hence T is a tree obtained from a bistar by subdividing the edge of maximum degree once.

Case 2. $d(u,v) \leq 2$ for all $v \in V(T)$

Suppose there exist two vertices v and w such that $d(u,v) = d(u,w) = 2$. Let P_1 be the $u-v$ path and P_2 be the $u-w$ path. Let $P_1 = (u, v_1, v)$ and $P_2 = (u, w_1, w)$. If $v_1 \neq w_1$ then $V - \{v_1, w_1\}$ is a nc2d-set of T which is a contradiction. If $v_1 = w_1$ then $V - \{u_1, v_1\}$ is a nc2d-set of T which is a contradiction. Hence at most one vertex of v such that $d(u,v) = 2$. Hence T is isomorphic to either star or $B(n-3,1)$.

The converse is obvious. □

Theorem 2.12. Let G be an unicyclic graph. Then $\gamma_{2nc}(G) = n-1$ if and only if G is isomorphic to C_3 or C_4 or $K_3(n_1, 0, 0)$ or $C_4(n_1, 0, n_2, 0)$, $n_1, n_2 \geq 1$.

Proof. Let G be an unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_r, v_1)$. If $G = C$ then by theorem 2.5, $G = C_3$ or C_4. Suppose $G \neq C$. Let A be the set of all pendant vertices in G. Clearly A is a subset of any γ_{2nc}-set of G.

Claim 1. Vertices of C of degree more than 2 or non adjacent.

Suppose not. Let v_i and v_j be the vertices of degree more than 2 in C. If v_i and v_j are adjacent then $V - \{v_i, v_j\}$ is a nc2d-set of G which is a contradiction.

Claim 2. $d(C, w) = 1$ for all $w \in A$

Suppose $d(C, w) \geq 2$ for some $w \in A$. Let $v_1, w_1, w_2, \cdots, w_k, w$ be the unique $v_1 - w$ path in G, $k \geq 1$. Then $V - \{w_1, v_2\}$ is a nc2d-set of G which is a contradiction. Hence $d(C, w) = 1$ for all $w \in A$.

Claim 3. $r \leq 4$

Suppose $r \geq 5$. Let $v_1 \in V(C)$ such that $\deg v_1 \geq 3$. Then $V - \{v_1, v_3\}$ is a nc2d-set of G which is a contradiction. Hence $r \leq 4$.

If $r = 3$ then G is isomorphic to $K_3(n_1, 0, 0)$. Let $r = 4$. Suppose $\deg v_1 \geq 3$ and $\deg v_i = 2$, $2 \leq i \leq 4$. Then $V - \{v_2, v_4\}$ is a nc2d-set of G which is a contradiction. Hence at least two vertices of C has degree greater than 2. Therefore G is isomorphic to $C_4(n_1, 0, n_2, 0)$, $n_1, n_2 \geq 1$.

The converse is obvious. □

Problem 2.13. Characterize the class of graphs for which $\gamma_{2nc}(G) = n-1$.

Theorem 2.14. Let G be a graph with $\delta(G) \geq 2$ then $\gamma_{2nc}(G) \leq 2\beta_1(G)$.

Proof. Let G be a graph with $\delta(G) \geq 2$ and M be a maximum set of independent edges in G. Let S be the vertices in the set of edges of M. Since $V - S$ is an independent set, each $v \in V - S$ must have at least two neighbors in S. Also since S contains no isolated vertices, $\langle N(S) \rangle$ is connected. Hence S is a nc2d-set of G. Thus $\gamma_{2nc}(G) \leq 2\beta_1(G)$. □

Problem 2.15. Characterize the class of graphs for which $\gamma_{2nc}(G) = 2\beta_1(G)$.

Theorem 2.16. Let G be a graph with $n \geq 4$. Then $\gamma_{2nc}(G) \geq \frac{2n+1-m}{2}$ and this bound is sharp.

Proof. Let S be a γ_{2nc}-set of G. Then each vertex of $V - S$ is adjacent to at least two vertices in S. If G is not a star then since $\langle N(S) \rangle$ is connected either $V - S$ or S contains at least one edge. Hence the number of edges $m \geq 2\,|V - S| + 1 = 2n - 2\gamma_{2nc} + 1$. Then $\gamma_{2nc} \geq \frac{2n+1-m}{2}$. The bound is sharp for C_5 and K_2. □

Problem 2.17. Characterize the class of graphs for which $\gamma_{2nc}(G) = \frac{2n+1-m}{2}$.

Theorem 2.18. For any graph G, $\gamma_{2nc}(G) \geq \frac{2n}{(\Delta+2)}$

Proof. Let S be a minimum nc2d-set and let k be the number of edges between S and $V-S$. Since the degree of each vertex in S is at most Δ, $k \leq \Delta\gamma_{2nc}$. But since each vertex in $V-S$ is adjacent to at least 2 vertices in S, $k \geq 2(n-\gamma_{2nc})$ combining these two inequalities produce $\gamma_{2nc}(G) \geq \frac{2n}{\Delta+2}$. □

Problem 2.19. Characterize the class of graphs for which $\gamma_{2nc}(G) = \frac{2n}{\Delta+2}$.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J. Paulraj Joseph and S. Arumugam [6] proved that $\gamma(G)+\chi(G) \leq n+1$. They also characterized the class of graphs for which the upper bound is attained. In following theorems we find the upper bound for the sum of the neighborhood connected 2-domination number and characterized the corresponding extremal graphs.

Theorem 2.20. For any graph G, $\gamma_{2nc}(G)+\chi(G) \leq 2n$ and equality holds if and only if G is isomorphic to K_2.

Proof. The inequality is obvious. Now we assume that $\gamma_{2nc}(G)+\chi(G) = 2n$. This implies $\gamma_{2nc}(G) = n$ and $\chi(G) = n$. Hence G is isomorphic to K_2. The converse is obvious. □

Theorem 2.21. Let G be a graph. Then $\gamma_{2nc}(G)+\chi(G) = 2n-1$ if and only if G is isomorphic to K_3.

Proof. Let us assume $\gamma_{2nc}(G)+\chi(G) = 2n-1$. This is possible only if (i) $\gamma_{2nc}(G) = n$ and $\chi(G) = n-1$ or (ii) $\gamma_{2nc}(G) = n-1$ and $\chi(G) = n$. Since the condition (i) is impossible, condition (ii) holds. This implies G is a complete graph with $\gamma_{2nc}(G) = n-1$. Then $n = 3$ and hence G is isomorphic to K_3. The converse is obvious. □

Theorem 2.22. For any graph G, $\gamma_{2nc}(G)+\chi(G) = 2n-2$ if and only if G is isomorphic to K_4 or P_3 or $K_3(1,0,0)$.

Proof. Let us assume $\gamma_{2nc}(G)+\chi(G) = 2n-2$. This is possible only if $\gamma_{2nc}(G) = n$ and $\chi(G) = n-2$ or $\gamma_{2nc}(G) = n-1$ and $\chi(G) = n-1$ or $\gamma_{2nc}(G) = n-2$ and $\chi(G) = n$.

Let $\gamma_{2nc}(G) = n$ and $\chi(G) = n-2$. Since $\gamma_{2nc}(G) = n$ which gives G is isomorphic to K_2 and hence $\chi(G) = 2 \neq n-2$ which is a contradiction.

Suppose $\gamma_{2nc}(G) = n-1$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, G contains a clique K on $n-1$ vertices. Let $V(K) = \{v_1, v_2, \cdots, v_{n-1}\}$ and $V(G) - V(K) = \{v_n\}$. Then v_n is adjacent to v_i for some vertex $v_i \in V(K)$. If $\deg(v_n) = 1$ and $n \geq 4$ then $\{v_i, v_j, v_n\}$, $i \neq j$ is a γ_{2nc}-set of G. Hence $n = 4$ and $K = K_3$. Thus G is isomorphic to $K_3(1,0,0)$. If $\deg v_n = 1$ and $n = 3$ then G is isomorphic to P_3. If $\deg(v_n) > 1$ then $\gamma_{2nc} = 2$. Then $n = 3$ which gives G is isomorphic to K_3 which is a contradiction to $\chi(G) = n-1$.

Suppose $\gamma_{2nc}(G) = n-2$ and $\chi(G) = n$. Since $\chi(G) = n$, G is isomorphic to K_n. But for K_n, $\gamma_{2nc}(G) = 2$. Therefore $n = 4$. Hence G is isomorphic to K_4. The converse is obvious. □

Theorem 2.23. Let G be a graph. Then $\gamma_{2nc}(G) + \chi(G) = 2n-3$ if and only if G is isomorphic to C_4 or $K_{1,3}$ or P_4 or K_5 or $K_3(2,0,0)$ or $K_4(1,0,0,0)$ or $K_4 - e$.

Proof. Let $\gamma_{2nc}(G) + \chi(G) = 2n-3$. This is possible only if $(i)\, \gamma_{2nc}(G) = n$, $\chi(G) = n-3$ or $(ii)\, \gamma_{2nc}(G) = n-1$, $\chi(G) = n-2$ or $(iii)\, \gamma_{2nc}(G) = n-2$, $\chi(G) = n-1$ or $(iv)\, \gamma_{2nc}(G) = n-3$, $\chi(G) = n$.

It is clear that (i) is impossible. Suppose (ii) holds. Since $\chi(G) = n-2$ G is either $C_5 + K_{n-5}$ or G contains a complete subgraph K on $n-2$ vertices. If $G = C_5 + K_{n-5}$ then $\gamma_{2nc}(G) + \chi(G) \neq 2n-3$. Thus G contains a complete subgraph K on $n-2$ vertices. Let $X = V(G) - V(K) = \{v_1, v_2\}$ and $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$.

Case 1. $\langle X \rangle = K_2$

Since G is connected, without loss of generality we assume v_1 is adjacent to v_3. If $|N(v_1) \cap N(v_2)| \geq 2$ then $\gamma_{2nc}(G) = 2$ and hence $n = 3$ which is a contradiction. So $|N(v_1) \cap N(v_2)| \leq 1$. Then $\{v_2, v_3, v_4\}$ is a γ_{2nc}-set of G and hence $n = 4$. If $|N(v_1) \cap N(v_2)| = 1$ then G is either $K_4 - e$ or $K_3(1,0,0)$. For these graphs $\chi(G) = 3$ which is a contradiction. If $N(v_1) \cap N(v_2) = \phi$. Then G is isomorphic to P_4 or C_4 or $K_3(1,0,0)$. Since $\chi[K_3(1,0,0)] = 3$, we have G is isomorphic to P_4 or C_4.

Case 2. $\langle X \rangle = \overline{K_2}$

Since G is connected v_1 and v_2 are adjacent to at least one vertex in K. If $\deg v_1 = \deg v_2 = 1$ and $N(v_1) \cap N(v_2) \neq \phi$ then $|V(K)| \neq 1$. So $|V(K)| \geq 2$. If $|V(K)| = 2$ then G is isomorphic to $K_{1,3}$. If $|V(K)| \geq 3$ then $\{v_1, v_2, v_3, v_4\}$ is a γ_{2nc}-set of G. Hence $n = 5$. Thus G is isomorphic to $K_3(2,0,0)$. If $\deg v_1 = \deg v_2 = 1$ and $N(v_1) \cap N(v_2) = \phi$ then $|V(K)| \geq 2$. If $|V(K)| = 2$ then G is isomorphic to P_4. If $V(K) \geq 3$ then $\{v_1, v_2, v_3, v_4\}$

is a γ_{2nc}-set of G. Hence $n = 5$. Thus G is isomorphic to $K_3(1, 1, 0)$. But $\gamma_{2nc}[K_3(1, 1, 0)] = 3$ which is a contradiction. Suppose $\deg v_1 \geq 2$ and $|N(v_1) \cap N(v_2)| \leq 1$ then $\{v_2, v_3, v_4\}$ where $v_3, v_4 \in N(v_1)$ is a γ_{2nc}-set of G. Hence $n = 4$. Then G is isomorphic to $K_3(1, 0, 0)$. For this graph $\gamma_{2nc}(G) = 3$ and $\chi(G) = 3$ which is a contradiction. If $\deg v_1 \geq 2$ and $N(v_1) \cap N(v_2) \geq 2$ then $\{v_3, v_4\}$ where $v_3, v_4 \in N(v_1) \cap N(v_2)$ is a γ_{2nc}-set of G. Then $n = 3$ which gives a contradiction.

Suppose (iii) holds. Since $\chi(G) = n - 1$, G contains a clique K on $n - 1$ vertices. Let $X = V(G) - V(K) = \{v_1\}$. If $\deg v_1 \geq 2$ then $\gamma_{2nc}(G) = 2$. Hence $n = 4$. Thus G is isomorphic to $K_4 - e$. If $\deg v_1 = 1$ then $|V(K)| \geq 3$ and hence $\{v_1, v_2, v_3\}$ where $v_2 \in N(v_1)$ is a γ_{2nc}-set of G and hence $n = 5$. Thus G is isomorphic to $K_4(1, 0, 0, 0)$.

Suppose (iv) holds. Since $\chi(G) = n$, G is a complete graph. Then $\gamma_{2nc}(G) = 2$ and hence $n = 5$. Therefore G is isomorphic to K_5. The converse is obvious. □

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