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A Note on the Primes in the Prime Factorization of an Integer

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Abstract

In this article we examine the primes that appear in the prime factorization of almost all numbers not exceeding x.

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1 Introduction

Let $k \geq 2$ be a fixed positive integer. Let $\alpha_k(x)$ be the set of numbers n not exceeding x such that in their prime factorization only appear primes p pertaining to the interval $\left[0, \frac{x}{k}\right]$. We assume that 1 pertains to the set $\alpha_k(x)$.

Let $\beta_k(x)$ be the set of numbers n not exceeding x such that in their prime factorization appear some prime p pertaining to the interval $\left(\frac{x}{k}, x\right]$.

Note that the sets $\alpha_k(x)$ and $\beta_k(x)$ are disjoints and $\alpha_k(x) \cup \beta_k(x) = A$, where A is the set of positive integers n such that $1 \leq n \leq \lfloor x \rfloor$.

Let $A_k(x)$ be the number of elements in the set $\alpha_k(x)$. Let $B_k(x)$ be the number of elements in the set $\beta_k(x)$. Consequently

$$A_k(x) + B_k(x) = \lfloor x \rfloor . \tag{1}$$

In this article we prove the asymptotic formula

$$A_k(x) \sim x. \tag{2}$$

Consequently almost all numbers $n \leq x$ have in their prime factorization only primes p pertaining to the interval $\left[0, \frac{x}{k}\right]$.

On the other hand, equations (1) and (2) imply the following formula

$$B_k(x) = o(x). \tag{3}$$

In this article we prove the more precise asymptotic formula

$$B_k(x) \sim B_k \frac{x}{\log x}, \tag{4}$$

where $B_k = 1/2 + 1/3 + \cdots + 1/k$.

Let $\pi(x)$ be the prime counting function. We shall need the well-known weak limit

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0, \tag{5}$$

and the more strong result (prime number Theorem)

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1. \tag{6}$$

Elementary proofs of limits (5) and (6) are given, for example, in [1, Chapter XXII].

2 Main Results

Theorem 2.1 *The following formula holds (see equation (3))*

$$B_k(x) = o(x).$$

Proof. Let $k \geq 2$ be a fixed positive integer. Consider the inequality

$$\frac{x}{k} < p \leq x, \tag{7}$$

where p denotes a positive prime number. Equation (7) implies that

$$pk > x.$$

Consequently the number of multiples of p not exceeding x is less than k.

On the other hand, the number of primes p that satisfy (7) is less than or equal to $\pi(x)$. Therefore

$$B_k(x) \leq k\pi(x). \tag{8}$$

Equations (8) and (5) imply that

$$\lim_{x\to\infty} \frac{B_k(x)}{x} = 0.$$

Equation (3) is proved. The theorem is proved.

Now, we prove the following more strong result.

Theorem 2.2 *The following formula holds (see equation (4))*

$$B_k(x) \sim B_k \frac{x}{\log x} \qquad (k \geq 2),$$

where $B_k = 1/2 + 1/3 + \cdots + 1/k$.

Proof. Let $k \geq 2$ be a fixed positive integer. Consider the inequality

$$\frac{x}{k} < p \leq x, \tag{9}$$

where p denotes a positive prime number.

If $x \geq k^2$ equation (9) gives $p > k$.

Consider the inequality

$$\frac{x}{2} < p \leq x. \tag{10}$$

The number of multiples of p not exceeding x is 1, namely p, since $p \leq x$ and $2p > x$.

Consequently the number of multiples of p not exceeding x such that p satisfies (10) is

$$\pi(x) - \pi(x/2). \tag{11}$$

Consider the inequality

$$\frac{x}{3} < p \leq \frac{x}{2}. \tag{12}$$

The number of multiples of p not exceeding x is 2, namely p and $2p$, since $2p \leq x$ and $3p > x$.

Consequently the number of multiples of p not exceeding x such that p satisfies (12) is

$$2\left(\pi(x/2) - \pi(x/3)\right). \tag{13}$$

$$\vdots$$

Consider the inequality

$$\frac{x}{k} < p \leq \frac{x}{k-1}. \tag{14}$$

The number of multiples of p not exceeding x is $k-1$, namely $p, 2p, \ldots, (k-1)p$, since $(k-1)p \leq x$ and $kp > x$. Note that p is the maximum prime factor in the prime factorization of the numbers $p, 2p, \ldots, (k-1)p$, since $p > k$ (see above).

Consequently the number of multiples of p not exceeding x such that p satisfies (14) is

$$(k-1)\left(\pi(x/(k-1)) - \pi(x/k)\right). \tag{15}$$

Therefore, see (11), (13), ..., (15), we have

$$\begin{aligned} B_k(x) &= (\pi(x) - \pi(x/2)) + 2\left(\pi(x/2) - \pi(x/3)\right) + 3\left(\pi(x/3) - \pi(x/4)\right) + \cdots \\ &+ (k-1)\left(\pi(x/(k-1)) - \pi(x/k)\right) = \pi(x) + \pi(x/2) + \pi(x/3) + \cdots \\ &+ \pi(x/(k-1)) - (k-1)\pi(x/k). \end{aligned} \tag{16}$$

Equation (6) implies

$$\lim_{x\to\infty}\frac{\pi(x/n)}{\pi(x)}=\frac{1}{n}. \tag{17}$$

Equations (16) and (17) give

$$\lim_{x\to\infty}\frac{B_k(x)}{\pi(x)}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k-1}-\frac{k-1}{k}=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}=B_k \tag{18}$$

Finally, equations (6) and (18) give equation (4). Equation (4) is proved. The theorem is proved.

Theorem 2.3 *The following asymptotic formula holds (see equation (2))*

$$A_k(x)\sim x.$$

Proof. It is an immediate consequence of equations (1) and (3). Equation (2) is proved. The theorem is proved.

In the former theorems we have considered the primes in the interval $\left[0,\frac{x}{k}\right]$. This interval has length $l(x)=x/k$. If k is large, this length $l(x)$ is little compared with x . That is, we have

$$\frac{l(x)}{x}=\frac{\frac{x}{k}}{x}=\frac{1}{k}.$$

We wish break the positive barrier $1/k$. Therefore, we shall consider an interval of length $l(x)$ such that

$$\lim_{x\to\infty}\frac{l(x)}{x}=0$$

and such that the former theorems (Theorem 2.1 and Theorem 2.3) hold.

Let $s>0$ be a fixed real number.

Let $\alpha_s(x)$ be the set of numbers n not exceeding x such that in their prime factorization only appear primes p pertaining to the interval $\left[0,\frac{x}{\log^s x}\right]$. Note that this interval has length $l(x)=\frac{x}{\log^s x}$ and

$$\lim_{x\to\infty}\frac{l(x)}{x}=0,$$

as we desired. That is, $l(x)=o(x)$.

Let $\beta_s(x)$ be the set of numbers n not exceeding x such that in their prime factorization appear some prime p pertaining to the interval $\left(\frac{x}{\log^s x},x\right]$.

Note that the sets $\alpha_s(x)$ and $\beta_s(x)$ are disjoints and $\alpha_s(x)\cup\beta_s(x)=A$, where A is the set of positive integers n such that $1\le n\le\lfloor x\rfloor$.

Let $A_s(x)$ be the number of elements in the set $\alpha_s(x)$. Let $B_s(x)$ be the number of elements in the set $\beta_s(x)$. Consequently

$$A_s(x) + B_s(x) = \lfloor x \rfloor . \tag{19}$$

Now, we shall prove two theorems analogous to Theorems 2.1 and 2.3. Before, we need the following lemma.

Lemma 2.4 *The following asymptotic formula holds*

$$\pi\left(\frac{x}{\log^h x}\right) = h(x)\frac{x}{\log^{1+h} x} \qquad (h > 0),$$

where $h(x) \to 1$.

Proof. Equation (6) is

$$\pi(x) = f(x)\frac{x}{\log x},$$

where $f(x) \to 1$. Therefore

$$\pi\left(\frac{x}{\log^h x}\right) = f\left(\frac{x}{\log^h x}\right)\frac{x}{\log^h x \log\left(\frac{x}{\log^h x}\right)} = h(x)\frac{x}{\log^{1+h} x},$$

where $h(x) \to 1$. The lemma is proved.

Theorem 2.5 *The following formula holds*

$$B_s(x) = o(x) \qquad (s > 0). \tag{20}$$

Proof. a) Let $0 < s < 1$ be a fixed real number. Consider the inequality

$$\frac{x}{\log^s x} < p \leq x, \tag{21}$$

where p denotes a positive prime number.

Equation (21) implies that $p\left(\lfloor \log^s x \rfloor + 1\right) > x$. Consequently the number of multiples of p not exceeding x is less than $\lfloor \log^s x \rfloor + 1$. On the other hand, the number of primes p that satisfy (21) is less than or equal to $\pi(x)$. Therefore

$$B_s(x) \leq \left(\lfloor \log^s x \rfloor + 1\right)\pi(x). \tag{22}$$

Equations (22) and (6) imply that

$$\lim_{x\to\infty} \frac{B_s(x)}{x} = 0 \qquad (0 < s < 1). \tag{23}$$

b) Let h' be an arbitrary but fixed real number such that $0 \leq h' < 1$. Consider the inequality

$$\frac{x}{\log^{1+h'} x} < p \leq x. \tag{24}$$

Let h be a fixed real number such that $0 < h < 1$ and $h > h'$. Inequality (24) can be divided in the form

$$\frac{x}{\log^{h} x} < p \leq x, \tag{25}$$

$$\frac{x}{\log^{1+h'} x} < p \leq \frac{x}{\log^{h} x}. \tag{26}$$

The number of multiples of p not exceeding x such that p satisfies inequality (25) is $o(x)$ (see part (a), equation (23)).

Equation (26) implies that $p\left(\left\lfloor \log^{1+h'} x \right\rfloor + 1\right) > x$. Consequently the number of multiples of p not exceeding x is less than $\left\lfloor \log^{1+h'} x \right\rfloor + 1$. On the other hand, the number of primes p that satisfy (26) is less than or equal to $\pi\left(\frac{x}{\log^h x}\right)$. Therefore (see lemma 2.4)

$$\begin{aligned} B_{1+h'}(x) &\leq o(x) + \left(\left\lfloor \log^{1+h'} x \right\rfloor + 1\right)\pi\left(\frac{x}{\log^h x}\right) \leq o(x) \\ &+ \left(1 + \log^{1+h'} x\right) h(x) \frac{x}{\log^{1+h} x} = o(x), \end{aligned}$$

since $h > h'$ (see above). Therefore if we put $s = 1 + h'$ then

$$\lim_{x\to\infty} \frac{B_s(x)}{x} = 0 \qquad (1 \leq s < 2). \tag{27}$$

c) Let n be a positive integer. We shall use mathematical induction. The theorem is true for $s = 1 + h$ $(0 \leq h < 1)$ (see part (b), equation (27)). Suppose that the theorem is true for $s = n + h$ $(0 \leq h < 1)$. We shall prove that the theorem is also true for $s = n + 1 + h'$ $(0 \leq h' < 1)$.

Let h' be an arbitrary but fixed real number such that $0 \leq h' < 1$. Consider the inequality

$$\frac{x}{\log^{n+1+h'} x} < p \leq x. \tag{28}$$

Let h be a fixed real number such that $0 < h < 1$ and $h > h'$. Inequality (28) can be divided in the form

$$\frac{x}{\log^{n+h} x} < p \leq x, \tag{29}$$

$$\frac{x}{\log^{n+1+h'} x} < p \leq \frac{x}{\log^{n+h} x}. \tag{30}$$

The number of multiples of p not exceeding x such that p satisfies inequality (29) is $o(x)$ (inductive hypothesis).

Equation (30) implies that $p\left(\left\lfloor \log^{n+1+h'} x\right\rfloor + 1\right) > x$. Consequently the number of multiples of p not exceeding x is less than $\left\lfloor \log^{n+1+h'} x\right\rfloor + 1$. On the other hand, the number of primes p that satisfy (30) is less than or equal to $\pi\left(\frac{x}{\log^{n+h} x}\right)$. Therefore (see lemma 2.4)

$$\begin{aligned} B_{n+1+h'}(x) &\leq o(x) + \left(\left\lfloor \log^{n+1+h'} x\right\rfloor + 1\right)\pi\left(\frac{x}{\log^{n+h} x}\right) \leq o(x) \\ &+ \left(1 + \log^{n+1+h'} x\right) h(x) \frac{x}{\log^{n+1+h} x} = o(x), \end{aligned}$$

since $h > h'$ (see above). Therefore if we put $s = n + 1 + h'$ then

$$\lim_{x\to\infty} \frac{B_s(x)}{x} = 0 \qquad (n + 1 \leq s < n + 2) \qquad (n \geq 1). \tag{31}$$

Equations (23), (27) and (31) imply equation (20). The theorem is proved.

Theorem 2.6 *The following asymptotic formula holds*

$$A_s(x) \sim x \qquad (s > 0).$$

Proof. It is an immediate consequence of equations (19) and (20). The theorem is proved.

If the interval has length very little compared with x these theorems are false.

For example, let $\alpha(x)$ be the set of numbers n not exceeding x such that in their prime factorization only appear primes p pertaining to the interval $[0, \log x]$. Note that this interval has lenght very little $l(x) = \log x$

Let $\beta(x)$ be the set of numbers n not exceeding x such that in their prime factorization appear some prime p pertaining to the interval $(\log x, x]$.

Note that the sets $\alpha(x)$ and $\beta(x)$ are disjoints and $\alpha(x) \cup \beta(x) = A$, where A is the set of positive integers n such that $1 \leq n \leq \lfloor x \rfloor$.

Let $A(x)$ be the number of elements in the set $\alpha(x)$. Let $B(x)$ be the number of elements in the set $\beta(x)$. Consequently

$$A(x) + B(x) = \lfloor x \rfloor . \tag{32}$$

Theorem 2.7 *the formula*

$$B(x) = o(x) \tag{33}$$

is false. Consequently the formula

$$A(x) \sim x \tag{34}$$

is also false.

Proof. A positive integer is quadratfrei if either is 1 or is a product of different primes. Let $Q(x)$ be the number of quadratfrei not exceeding x. We have [1, Chapter XVIII]

$$Q(x) = f(x)x,$$

where $f(x) \to \frac{6}{\pi^2}$.

If equation (33) holds then the number $Q_1(x)$ of quadratfrei such that in their prime factorization only appear primes p pertaining to the interval $[0, \log x]$ is

$$Q_1(x) = Q(x) - o(x) = f(x)x - o(x) = f_1(x)x,$$

where $f_1(x) \to \frac{6}{\pi^2}$.

On the other hand, the number of all possible quadratfrei such that in their prime factorization only appear primes p pertaining to the interval $[0, \log x]$ is

$$\binom{\pi(\log x)}{0} + \binom{\pi(\log x)}{1} + \binom{\pi(\log x)}{2} + \cdots + \binom{\pi(\log x)}{\pi(\log x)} = 2^{\pi(\log x)}.$$

Consequently we have

$$2^{\pi(\log x)} \geq Q_1(x) = f_1(x)x.$$

That is

$$\pi(\log x) \log 2 \geq f_2(x) \log x,$$

where $f_2(x) \to 1$.

Therefore

$$\pi(\log x) \geq f_3(x) \log x > h \log x, \tag{35}$$

where $f_3(x) \to \frac{1}{\log 2}$ and $\frac{1}{\log 2} > h > 1$.

Now, equation (35) is an evident contradiction since $\pi(x) \leq x$. Consequently equation (33) is false. The theorem is proved.

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References

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