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Sum of the Reciprocal of the Primes in the Prime Factorization of $n!$

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In memory of my sister Fedra Marina Jakimczuk (1970-2010).

Abstract

Let $\alpha_m(n)$ be the sum of the reciprocal of the m-th powers of the different primes in the prime factorization of n. We prove the asymptotic formula

$$\sum_{i=2}^{n} \alpha_m(i) = A_m\ n + o(n),$$

where A_m is a constant defined in this article.

Let $\beta_m(n)$ be the sum of the reciprocal of the m-th powers of the primes in the prime factorization of n. We prove the asymptotic formula

$$\sum_{i=2}^{n} \beta_m(i) = B_m\ n + o(n),$$

where B_m is a constant defined in this article.

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1 Preliminary Results

Let $N_p(n)$ be the number of primes p in the prime factorization of n. For example $N_p(1) = 0$, $N_2(24) = N_2(2^3.3) = 3$, $N_3(125) = N_3(5^3) = 0$ and $N_p(p^k) = k$. Clearly, the following equation holds

$$\sum_{i=1}^{n} N_p(i) = N_p(n!). \tag{1}$$

The following Legendre's result is well-known (see [1], pages 90-91)

$$\sum_{i=1}^{n} N_p(i) = N_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ denotes the integer part of x. Note that

$$0 \leq x - \lfloor x \rfloor < 1.$$

We shall need the following well-known formula

$$1 + x + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} \qquad (x \neq 1). \tag{3}$$

We also have the following result (see [1], page 95)

$$\sum_{i=1}^{n} \frac{1}{i} \sim \log n. \tag{4}$$

Let $\pi(x)$ be the prime counting function. We shall need the following well-known weak equation

$$\pi(x) = o(x). \tag{5}$$

Besides, we shall need the following lemma.

Lemma 1.1 *The following formula holds (see (2))*

$$\sum_{i=1}^{n} N_p(i) = N_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n}{p-1} - r_p(n) - f_p(n) \qquad (n \geq p), \tag{6}$$

where

$$\frac{1}{p-1} \leq r_p(n) \leq \frac{p}{p-1},$$

$$0 \leq f_p(n) \leq \frac{\log n}{\log p}.$$

Proof. If $n \geq p$ then the inequality

$$p^k \leq n$$

has the solutions

$$k = 1, 2, \ldots, s_n = \left\lfloor \frac{\log n}{\log p} \right\rfloor. \tag{7}$$

Therefore we have (see (7))

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k=1}^{s_n} \left\lfloor \frac{n}{p^k} \right\rfloor = n \sum_{k=1}^{s_n} \frac{1}{p^k} - f_p(n), \tag{8}$$

where

$$0 \leq f_p(n) \leq \frac{\log n}{\log p}. \tag{9}$$

Note that (see (3))

$$\sum_{k=1}^{s_n} \frac{1}{p^k} = \frac{1}{p} \sum_{k=0}^{s_n-1} \frac{1}{p^k} = \frac{1}{p} \frac{1-\left(\frac{1}{p}\right)^{s_n}}{1-\frac{1}{p}} = \frac{1-\left(\frac{1}{p}\right)^{s_n}}{p-1} = \frac{1}{p-1} - \frac{1}{p-1}\frac{1}{p^{s_n}}. \tag{10}$$

Now, we have

$$\frac{1}{p^{s_n}} = \frac{1}{p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor}} = \frac{1}{p^{\left(\frac{\log n}{\log p} - g_n\right)}} = \frac{p^{g_n}}{n}, \tag{11}$$

where

$$0 \leq g_n \leq 1. \tag{12}$$

Consequently (see (11) and (12))

$$\frac{1}{n} \leq \frac{1}{p^{s_n}} \leq \frac{p}{n}. \tag{13}$$

Finally, equations (8), (9), (10) and (13) give equation (6). The lemma is proved.

2 Main Results

Let $\alpha_m(n)$ be the sum of the reciprocal of the m-th powers of the different primes in the prime factorization of n, where $m \geq 1$ is a positive integer. For example $\alpha_1(2^3.5^2.11) = \frac{1}{2} + \frac{1}{5} + \frac{1}{11}$ and $\alpha_3(2^3.5^2.11) = \frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{11^3}$.

We have the following theorem.

Theorem 2.1 *The following asymptotic formula holds*

$$\sum_{i=2}^{n} \alpha_m(i) = A_m\ n + o(n), \tag{14}$$

where

$$A_m = \sum_p \frac{1}{p^{m+1}}.$$

Proof. We have

$$\sum_{i=2}^{n} \alpha_m(i) = \sum_{p \leq n} \frac{1}{p^m} \left\lfloor \frac{n}{p} \right\rfloor = \sum_{p \leq n} \frac{1}{p^m} \left(\frac{n}{p} - \epsilon_p(n) \right) = n \sum_{p \leq n} \frac{1}{p^{m+1}} - \sum_{p \leq n} \frac{\epsilon_p(n)}{p^m} \tag{15}$$

where $0 \leq \epsilon_p(n) < 1$.

Note that

$$\sum_{p \leq n} \frac{1}{p^{m+1}} = \sum_{p} \frac{1}{p^{m+1}} + o(1). \tag{16}$$

Besides (see (4))

$$0 \leq \sum_{p \leq n} \frac{\epsilon_p(n)}{p^m} \leq \sum_{p \leq n} \frac{1}{p^m} \leq \sum_{p \leq n} \frac{1}{p} \leq \sum_{i=1}^{n} \frac{1}{i} = o(n).$$

Therefore

$$\sum_{p \leq n} \frac{\epsilon_p(n)}{p^m} = o(n). \tag{17}$$

Equations (15), (16) and (17) give equation (14). The theorem is proved.

Let $\beta_m(n)$ be the sum of the reciprocal of the m-th powers of the primes in the prime factorization of n, where $m \geq 1$ is a positive integer. For example $\beta_1(2^3.5^2.11) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{11}$ and $\beta_3(2^3.5^2.11) = \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{11^3}$.

We have the following theorem.

Theorem 2.2 *The following asymptotic formula holds*

$$\sum_{i=2}^{n} \beta_m(i) = B_m \ n + o(n), \tag{18}$$

where

$$B_m = \sum_{p} \frac{1}{p^m(p-1)}.$$

Proof. We have (see (2) and lemma 1.1)

$$\begin{aligned} \sum_{i=2}^{n} \beta_m(i) &= \sum_{p \leq n} \frac{1}{p^m} \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \right) = \sum_{p \leq n} \frac{1}{p^m} \left(\frac{n}{p-1} - r_p(n) - f_p(n) \right) \\ &= n \sum_{p \leq n} \frac{1}{p^m(p-1)} - \sum_{p \leq n} \frac{1}{p^m} r_p(n) - \sum_{p \leq n} \frac{1}{p^m} f_p(n). \end{aligned} \tag{19}$$

Note that

$$\sum_{p \leq n} \frac{1}{p^m(p-1)} = \sum_{p} \frac{1}{p^m(p-1)} + o(1). \tag{20}$$

We have (see lemma 1.1)

$$\frac{1}{p-1} \leq r_p(n) \leq \frac{p}{p-1}.$$

Consequently

$$0 \leq \frac{1}{p^m}\frac{1}{p-1} \leq \frac{1}{p^m} r_p(n) \leq \frac{1}{p^{m-1}}\frac{1}{p-1} \leq \frac{1}{p-1} \leq 1.$$

That is

$$0 \leq \frac{1}{p^m} r_p(n) \leq 1.$$

Therefore (see (5))

$$0 \leq \sum_{p \leq n} \frac{1}{p^m} r_p(n) \leq \pi(n) = o(n).$$

That is

$$\sum_{p \leq n} \frac{1}{p^m} r_p(n) = o(n). \tag{21}$$

We have (see lemma 1.1)

$$0 \leq f_p(n) \leq \frac{\log n}{\log p}.$$

Consequently

$$0 \leq \frac{1}{p^m} f_p(n) \leq \frac{1}{p^m}\frac{\log n}{\log p} \leq \frac{1}{p \log p} \log n.$$

Therefore (see (4))

$$\begin{aligned} 0 \quad &\leq \quad \sum_{p \leq n} \frac{1}{p^m} f_p(n) \leq \log n \sum_{p \leq n} \frac{1}{p \log p} \leq \log n \sum_{i=2}^{n} \frac{1}{i \log i} \leq \log n \sum_{i=2}^{n} \frac{\log i}{\log 2}\frac{1}{i \log i} \\ &\leq \quad \frac{\log n}{\log 2} \sum_{i=1}^{n} \frac{1}{i} = o(n). \end{aligned}$$

That is

$$\sum_{p \leq n} \frac{1}{p^m} f_p(n) = o(n). \tag{22}$$

Equations (19), (20), (21) and (22) give equation (18). The theorem is proved.

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References

[1] W. J. LeVeque, *Topics in Number Theory,* Volume 1, Addison-Wesley, First Edition, 1958.

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