International Mathematical Forum, Vol. 7, 2012, no. 39, 1941 - 1948

A Note on the Nonlinear Recurrence $x_{n+1}x_{n-1} - x_n^2 = A$

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Abstract

The existence and uniqueness of the integral solutions of $x_{n+1}x_{n-1} - x_n^2 = A$ are examined and some open questions settled.

Mathematics Subject Classification: 39A99, 12F05, 11B39

1 Introduction

The sequences $\mathcal{X} = \{x_1, x_2, \ldots, x_n, \ldots\}$ satisfying the nonlinear recurrence

$$x_{n+1}x_{n-1} - x_n^2 = A \quad , \tag{1}$$

with $A \neq 0$ and initial values x_1, x_2 specified, have been extensively considered, together with related linear sequences [1, 4, 5, 6, 11]. In particular, Alperin in [1] looked for integral sequence solutions of (1), i.e. sequences of integer numbers, and asked for which integer values of A these sequences are unique, except for shifts and sign changes. This question is apparently still open, and its investigation is the main concern of this paper. We recall from [1] two peculiar properties of the sequences satisfying (1), that is

1) any sequence $\mathcal{X}$ satisfying (1) also satisfies a linear recurrence $x_{n+1} = \mu x_n - x_{n-1}$ of the second order, with μ specified by A, x_1, and x_2, [1, Proposition 2.1];

2) a sequence $\mathcal{X}$ satisfying (1) is integral if the equations

$$X^2 - \frac{\mu^2-4}{4}Y^2 = -A, \quad \mu \ \text{ even}, \quad \text{or} \quad X^2-(\mu^2-4)Y^2 = -A, \quad \mu \ \text{ odd}$$

have even discriminant; if the discriminant is odd, then necessarily μ is odd and the equation $X^2-(\mu^2-4)Y^2 = -4A$ has a solution with X odd [1, Theorem 3.1].

Using the notion of derived sequence, these and other properties may be proved in a way suitable to tackle the uniqueness problem of integral solutions. The first derived sequence $\mathcal{Y} = \{y_1, y_2, \ldots, y_n, \ldots\}$ of a recurring sequence $\mathcal{X}$ is defined, [4], as

$$y_n = x_n^{(1)} = \begin{vmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{vmatrix} = x_{n+1}x_{n-1} - x_n^2 \quad . \tag{2}$$

If $y_n = A$, then $\mathcal{X}$ satisfies the nonlinear recurrence (1), and obviously $\mathcal{Y}$ satisfies the first-order recurrence $y_{n+1} = y_n$ with initial condition $y_1 = A$. It is a general property of derived sequences that the first derived sequence satisfies a linear recurrence if and only if the original sequence satisfies a second-order recurrence [5, Theorem 1]. It follows that $\mathcal{X}$ must satisfy a second-order linear recurrence, i.e. $x_{n+1} = cx_n + dx_{n-1}$, with derived sequence $y_n = -x_{n+1}^2 + cx_{n+1}x_n - x_n^2$ (the right-side expression is known as the Simson formula) satisfying the recurrence $y_{n+1} = -dy_n$. Then, identifying the coefficients of the two equations satisfied by y_n, we have

$$d=-1 \quad , \qquad c=\mu \quad , \qquad A = y_1 = x_2x_0 - x_1^2 \quad , \qquad \text{and} \qquad x_2 = \mu x_1 - x_0 \quad .$$

Consequently, A is represented by the principal quadratic form $A = -x_0^2 + \mu x_0 x_1 - x_1^2$, of discriminant μ^2-4. These conclusions are summarized in the following theorem, which may be considered a rephrasing of [1, Theorem 3.1].

Theorem 1. *The sequence $\mathcal{X}$ defined by equation (1), given the integers A and μ, is integral if and only if the principal quadratic form $Q(x,y) = -x^2+\mu xy-y^2$ represents A, and the initial values x_1 and x_2 are integers such that $Q(x_1,x_2) = A$.*

A necessary condition for A being properly represented by $Q(x,y)$, of discriminant μ^2-4, is that this discriminant be a quadratic residue modulo every prime factor of A [2, 3]. However, necessary and sufficient conditions for A to be represented by the principal form $Q(x,y)$ are more difficult to establish. An idea of the complexity is offered by a sufficient condition that is not difficult to prove. To this aim, observe that $Q(x,y)$ is equivalent to the forms

$$Q'(x,y) = -x^2 + \frac{\mu^2-4}{4}y^2 \quad \text{or} \quad Q'(x,y) = -x^2 + xy + \frac{\mu^2-5}{4}y^2 \ ,$$

if μ is even or odd, respectively.

Theorem 2. *Given a square-free $A > 0$, and fixing μ, a sufficient condition for A being (properly)represented by $Q(x,y) = -x^2 + \mu xy - y^2$ is that every prime factor p_i of A, or its negative $-p_i$, is represented by $Q(x,y)$, and that the number of positive primes represented is odd.*

Proof. Since the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{\mu^2-4})$ has field norm 1, the quadratic form $Q(x,y)$ may represent either p, or $-p$, but not both. Let $A = \prod_{i=1}^k p_i$ be a product of k distinct primes. Assuming that the number μ is even, the quadratic form $Q(x,y)$ is equivalent to a reduced form $q(X,Y) = -X^2 + \frac{\mu^2-4}{4}Y^2$, then every prime p_i, or its negative, represented by $Q(x,y)$ splits in $\mathbb{Q}(\sqrt{\mu^2-4})$ as

$$p_i = (-1)^{\nu(i)} \left(a_{p_i} + b_{p_i}\sqrt{\frac{\mu^2-4}{4}} \right) \left(a_{p_i} - b_{p_i}\sqrt{\frac{\mu^2-4}{4}} \right) \ ,$$

where $\nu(i)$ is 1 if p_i is represented by $q(X,Y)$ and $\nu(i)$ is 0 if $-p_i$ is represented by $q(X,Y)$. Therefore, $A > 0$ splits as

$$A = \prod_i p_i = \prod_i (-1)^{\nu(i)} \prod_i \left(a_{p_i} + b_{p_i}\sqrt{\frac{\mu^2-4}{4}} \right) \prod_i \left(a_{p_i} - b_{p_i}\sqrt{\frac{\mu^2-4}{4}} \right) \ .$$

It follows that

$$A = \left(A_0 + B_0\sqrt{\frac{\mu^2-4}{4}} \right) \left(A_0 - B_0\sqrt{\frac{\mu^2-4}{4}} \right) \prod_i (-1)^{\nu(i)} \ .$$

Then, $A > 0$ is represented by $q(X,Y)$ if and only if $\prod_i (-1)^{\nu(i)} = -1$, i.e. the number of positive primes represented by $q(X,Y)$ is odd. The same argument holds for odd μ with a_p replaced by $a_p - \frac{b_p}{2}$ and b_p by $\frac{b_p}{2}$, and the conclusion is the same.

□

2 Uniqueness

Given A and μ, let (x_1, x_2) be a proper solution of $A = -x^2 + \mu xy - y^2$, then we have four sequences (two if $x_1 = x_2$) satisfying the recurrence $x_{n+1} = ax_n - x_{n-1}$, which correspond to the initial conditions

$$(x_1, x_2), (-x_1, -x_2) \quad , \quad (x_2, x_1), (-x_2, -x_1) \quad .$$

This is because $\mu = \frac{x_1^2+x_2^2+A}{x_1 x_2}$ is a symmetric function of x_1 and x_2 that remains invariant when the signs of both variables are changed. Furthermore, given A, if (x_1, x_2) is a solution of $A = -x^2 + \mu xy - y^2$, then $(-x_1, x_2)$ is a solution of $A = -x^2 - \mu xy - y^2$. Therefore, for each sequence satisfying (1) with initial condition (x_1, x_2), there are eight sequences (or four if $|x_1| = |x_2|$) satisfying the same recurrence. Since, for any given $|A| \geq 3$, a solution certainly exists which corresponds to $\mu = A + 2$, and $x_1 = x_2 = 1$, uniqueness is defined as follows.

Definition 1. *An integer A uniquely identifies a class of four sequences satisfying (1), if it specifies a unique absolute value $|\mu| = |A + 2|$.*
Equivalently, A uniquely identifies a class of four sequences satisfying (1) if it is represented by a quadratic form of the type $-x^2 + \mu xy - y^2$, with unique $|a|$.

Let $N(A)$ be the number of quadratic forms $-x^2 + \mu xy - y^2$ that represent A. Given $|A| \geq 3$, the coefficient $\mu = A + 2$, that corresponds to the initial values $x_1 = x_2 = 1$, certainly identifies a sequence satisfying (1). Thus, a representation of A is always given by $(1, 1)$, and we have at least four sequences that represent A

$$\begin{array}{rrrr} \ldots, A+1, & 1, & 1, & A+1, \ldots \\ \ldots, -A-3, & -1, & 1, & A+3, \ldots \\ \ldots, A+3, & 1, & -1, & -A-3, \ldots \\ \ldots, -A-1, & -1, & -1, & -A-1, \ldots \end{array} \quad . \tag{3}$$

In the search for A's that admit a unique representation, we should ascertain that the only pairs representing A are pairs of consecutive numbers in some of these four sequences. Therefore, besides the sequences (3) related to the quadratic form $-x^2 + (A + 2)xy - y^2$, we must look for any other quadratic form $-x^2 + \mu xy - y^2$ representing A. In this last case, μ must satisfy the necessary condition that $\mu^2 - 4$ is a quadratic residue for A. This quadratic residuosity condition is also a sufficient condition if the class number of the field $\mathbb{Q}(\sqrt{\mu^2 - 4})$ is 1, otherwise, if the class number is greater than 1, we have more that one class (or genus) of quadratic forms, then further conditions should be satisfied in order for A to be represented by a principal form. This problem is already difficult when $A = p$ is a prime [3]; however, for some particular values of A we have definitive answers:

$A = 1$: there is a unique $\mu = 3$ such that $-x^2 + \mu xy - y^2$ represents 1 as proved in [1].

$A = -1$: the number of μ's is clearly infinite: the reason, already given in [1], is that the Pell equation $x^2 - Dy^2 = 1$, $D > 1$, is always solvable, thus $-1 = -x^2 + \mu\, xy - y^2$ is solvable for any μ.

$A = \pm 2$**:** it will be seen below that there is no μ such that $-x^2 + \mu xy - y^2$ represents ± 2.

$|A| > 1$: the number of μ's such that the equation $A = -x^2 + \mu\, xy - y^2$ is solvable is finite as shown below.

In the proof of the following theorems, we need the continued fraction expansions of $\sqrt{\mu^2 - 4}$ and some related properties [7, pages 262-265]:

1. odd μ
$$\sqrt{\mu^2-4} = [\mu - 1, \quad \overline{1, \frac{\mu-3}{2}, 2, \frac{\mu-3}{2}, 1, 2\mu - 2}] \quad ,$$
2. even μ
$$\sqrt{\mu^2-4} = [\mu - 1, \quad \overline{1, \frac{\mu}{2} - 2, 1, 2(\mu - 1)}] \quad .$$

Let $\frac{p_n}{q_n}$ denote a convergent, and define the sequence $\Delta_n = p_n^2 - (\mu^2-4)q_n^2$. If L is the period of the continued fraction, then $\Delta_{L-1} = (-1)^L$ and $x = p_{L-1}$, $y = q_{L-1}$ is the minimal solution of the Pell equation $x^2 - (\mu^2-4)y^2 = (-1)^L$. The negative Pell equation has no solution when L is even. In our case, with the exception of $\mu = 3$ when $L = 1$, L is always even, then $-x^2 + \mu xy - y^2 = 1$ has no solutions if $|\mu| > 3$. Since, for both even and odd μ, the periods are even, the sequences Δ_n associated to the convergent $\frac{p_n}{q_n}$ are

$$4, -\mu + 2, 4, -2\mu + 5, 4, -2\mu + 5 \quad \text{and} \quad 4, -2\mu + 5, 1, -2\mu + 5 \quad ,$$

respectively. These sequences give all numbers δ in absolute value less than $\sqrt{\mu^2 - 4}$ such that $x^2 - (\mu^2-4)y^2 = \delta$ is solvable [7, Theorem 8.2]. Therefore, their negatives are the only numbers of absolute value less than $\sqrt{\mu^2 - 4}$ which are representable by $-x^2 + \mu xy - y^2$, that is the numbers -1, -4 , $\mu - 2$ and $2\mu - 5$. It follows that $A = \pm 2$ cannot be represented by any quadratic form.

Theorem 3. *If an integer A is represented by the quadratic form $-x^2+\mu xy-y^2$, the absolute value of μ is bounded as $|\mu| \leq |A|+2$; consequently, the number of quadratic forms $-x^2+\mu xy-y^2$ that represent a given A is finite.*

PROOF. Given A, then it is certainly represented by $-x^2+\mu xy-y^2$ with $\mu = A+2$. A solution is $x=1, y=1$, the discriminant is $\mu^2-4 = A(A+2)$, $A > 0, A < -2$, and the prime factors of A ramify in the quadratic field $\mathbb{Q}(\sqrt{A(A+2)})$.
From the discussion preceding the theorem, for any given μ, the only representable numbers A of absolute value less than μ are -1, -4 , $\mu-2$ and $2\mu-5$. It follows that, if $\mu > 0$, the only representable number A is $\mu-2$, besides the three numbers $1,3,5$ which are of the form $2\mu-5$ for $\mu = 3,4,5$ respectively. Whereas, if $\mu < 0$ there are no representable numbers less than $|\mu|$. For a given μ, if $A < \mu$ then only the number $\mu-2$ is representable. It follows that for a given A the absolute value of the number μ must be smaller than $|A|+2$, then the number of μ's, and thus of quadratic forms, representing A is finite.

□

The number $N(A)$ of quadratic forms $-x^2+\mu xy-y^2$ that represent A is greater than or equal 1, and the following theorems show that it is frequently greater than 1.

Theorem 4. *If the integer A is not of the form $p-1$, with p prime, there is a quadratic form $-x^2+\mu xy-y^2$ with $\mu \neq A+2$ that represents A with $y=1$ and x a root of the quadratic equation $-x^2+\mu x-1 = A$.*

PROOF. Writing the equation $-x^2+\mu x-1 = A$ in the form $x(\mu-x) = A+1$, since $A+1$ is not prime there is at least a factorization $A+1 = \alpha_1\alpha_2$ with both α_1 and α_2 greater than 1. Therefore the solution $x = \alpha_1$ and $\mu = \alpha_2+\alpha_1$ settles the question. Actually A is represented as $(\alpha_1, 1)$ by the form with $\mu = \alpha_2+\alpha_1$.

□

Theorem 5. *If $A = p-1$, where p is a prime of the form $4k+1$ or $2(2h+1)^2k+1$, there exists at least a second representation $(2,2)$ with $\mu = A/4+2$, and $(2h+1, 2h+1)$ with $\mu = 2(2h+1)(k+1)$.*

PROOF. If $A = p-1 = 4k$, writing the equation $-x^2+2\mu x-4 = 4k$ in the form $x(2\mu-x) = 4(k+1)$, then taking $x=2$ and $\mu = k+2 = \frac{A}{4}+2$ we have a representation.

If $A = p-1 = 2(2h+1)^2k$, writing the equation $-x^2+(2h+1)\mu x-(2h+1)^2 = 2(2h+1)^2k$ in the form $x((2h+1)\mu - x) = (2h+1)^2(2k+1)$, we certainly have the solution $x = 2h+1$ and $\mu = 2(2h+1)(k+1)$.

□

3 Conclusions

In summary, concerning uniqueness, Theorems 4 and 5 only leave open some cases when $A = p-1$ and p is congruent 3 modulo 4. In this case, an exhaustive analysis, considering all primes $p = A+1 = 3 \bmod 4$ less than 200, showed that $N(A) \geq 2$ for the following values of A

$$18, 66, 126, 138, 150, 162, 198 \quad ,$$

and $N(A) = 1$ for the following values of A

$$6, 10, 22, 30, 42, 46, 58, 70, 78, 82, 102, 106, 130, 166, 178, 190 \quad ,$$

this second list should be completed with the addition of $A = 1$.
Obviously, one of the two lists certainly extends to infinity; however, it is likely that both lists are unlimited. Curiously, the second list includes every $A < 200$ such that $A+1$ is a Sophie Germain prime q_{sg}. This observation together with the fact that every checked (randomly chosen) q_{sg} had $N(q_{sg}+1) = 1$, supports a guess that the second list includes every Sophie Germain prime.

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Received: March, 2012