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The k-Bessel Function of the First Kind

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Abstract

In this short paper we introduce a version k of Bessel function of first kind. We study some basic properties and present a relationship linking this function with the functions k-Mittag-Leffler and k-Wright recently introduced by autors (cf.[3], [6]).

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I Introduction and Preliminaries

What can be called the k-calculus began with the definition made by Diaz and Pariguan (cf.[2]) of the k-gamma function and the Pochhammer k-symbol as generalizations of the known functions the classical gamma function and the classical Pochhammer symbol. Since then there are many works devoted to generalizations of known special functions related to the fractional calculus as well as of fractional integral operators. Start recalling some definitions of elements that well be used in developing this paper.

In this way they have

Definition 1 *Let k be a positive real number. The k-gamma function is given by*

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \; Re(z) > 0. \;\; cf.[2] \tag{I.1}$$

We observe that $\Gamma_k(z)$ reduces to the classical $\Gamma(z)$ function when $k = 1$.

Definition 2

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad \gamma \in \mathbb{C}, k \in \mathbb{R} \text{ and } n \in \mathbb{N} \qquad \text{(I.2)}$$

Among the special functions related to the fractional calculus we point out the k-Mittag-Leffler function and the k-Wright function given by the following

Definition 3 *Let $k \in \mathbb{R}$; $\alpha, \beta, \gamma \in \mathbb{C}$; $Re(\alpha) > 0, Re(\beta) > 0$, the k-Mittag-Leffler function is defined by the following serie*

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}. \quad (cf.[3]). \qquad \text{(I.3)}$$

Definition 4 *Let $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > -1$, $Re(\beta) > 0$, $k \in \mathbb{R}$, $n \in \mathbb{N}$. The k-Wright function is defined for*

$$W^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(n!)^2}. \quad (cf.[6]). \qquad \text{(I.4)}$$

where $(\gamma)_{n,k}$ is the k-Pochhammer symbol given in (I.2) and $\Gamma_k(x)$ is the k-gamma function given in (I.1).

For our purpose we need also the following

Definition 5 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ an exponential order function and piecewise continuous, then the Laplace transform of f is*

$$\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t) dt. \qquad \text{(I.5)}$$

The integral exist for $Re(s) > 0$. Cf. [1] p. 40.

II Main results

In this section we introduce a new Bessel type function in the context of the k-calculus and consider some of their properties and the action of Laplace transform on it. That result an important relationship between k-Mittag-Leffler (cf.[3]), k-Wright function (cf.[6]) and the k-Bessel of first kind.

Definition 6 *Let $k \in \mathbb{R}$; $\alpha, \lambda, \gamma, \nu \in \mathbb{C}$; $Re(\lambda) > 0, Re(\nu) > 0$, the k-Bessel function of the first kind is defined by the following serie*

$$J^{(\gamma)(\alpha)}_{k,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-1)^n (z/2)^n}{(n!)^2} \qquad \text{(II.1)}$$

where $(\gamma)_{n,k}$ is the k-Pochhammer symbol given in (I.2) and $\Gamma_k(x)$ is the k-gamma function given in (I.1).

From the definition of k-Wright function (I.4) result that

$$J_{k,\nu}^{(\gamma)(\lambda)}(z) = \left(\frac{z}{2}\right)^{\nu} W_{k,\lambda,\nu+1}^{\gamma}\left(-\frac{z^2}{4}\right) \tag{II.2}$$

It may be observed that $J_{k,\nu}^{(\gamma)(\lambda)}(z)$ is such that $J_{k,\nu}^{(\gamma)(\lambda)}(z) \longrightarrow J_{\nu}^{(\lambda)}(z)$ as $k \to 1$ and $\lambda \to 1$, since $(\gamma)_{n,k} \longrightarrow (\gamma)_n$, $\Gamma_k(z) \longrightarrow \Gamma(z)$ and the convergence of the series in (II.1) is uniform on compact subsets.

Lemma 1 *Let* $\alpha, \gamma, \nu \in \mathbb{C}$, $Re(\alpha) > 0, Re(\nu) > 0$. *Then*

$$\left(\frac{\gamma}{k}\right) J_{k,\nu}^{(\gamma+k)(\alpha)}(z) - \left(\frac{2\gamma - k\nu}{2k}\right) J_{k,\nu}^{(\gamma)(\alpha)}(z) = \frac{z}{2}\frac{d}{dz} J_{k,\nu}^{(\gamma)(\alpha)}(z) \tag{II.3}$$

Proof. We known that

$$W_{k,\alpha,\beta}^{\gamma+k}(z) - W_{k,\alpha,\beta}^{\gamma}(z) = \left(\frac{k}{\gamma}\right) z \frac{d}{dz} W_{k,\alpha,\beta}^{\gamma}(z) \quad cf[6] \tag{II.4}$$

then

$$W_{k,\alpha,\nu+1}^{\gamma+k}\left(-\frac{z^2}{4}\right) - W_{k,\alpha,\nu+1}^{\gamma}\left(-\frac{z^2}{4}\right) = \left(\frac{k}{\gamma}\right) \frac{z}{2}\frac{d}{dz} W_{k,\alpha,\nu+1}^{\gamma}\left(-\frac{z^2}{4}\right) \tag{II.5}$$

multiplying both sides by $\left(\frac{z}{2}\right)^{\nu}$ and using (II.2) result

$$J_{k,\nu}^{(\gamma+k)(\alpha)}(z) - J_{k,\nu}^{(\gamma)(\alpha)}(z) = \left(\frac{z}{2}\right)^{\nu} \left(\frac{k}{\gamma}\right) \frac{z}{2}\frac{d}{dz} W_{k,\alpha,\nu+1}^{\gamma}\left(-\frac{z^2}{4}\right) \tag{II.6}$$

Taking into account that

$$\frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] = \frac{d}{dz}\left[\left(\frac{z}{2}\right)^{\nu} W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right)\right] \tag{II.7}$$

$$= \left(\frac{1}{2}\right)^{\nu} z^{\nu-1} \nu W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right) + \left(\frac{z}{2}\right)^{\nu} \frac{d}{dz} W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right) \tag{II.8}$$

we have

$$\left(\frac{z}{2}\right)^{\nu} \frac{d}{dz} W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right) = \frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] - \left(\frac{1}{2}\right)^{\nu} z^{\nu-1} \nu W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right) \tag{II.9}$$

Then, from (II.6) and (II.9) result

$$\begin{aligned} J_{k,\nu}^{(\gamma+k)(\alpha)}(z) - J_{k,\nu}^{(\gamma)(\alpha)}(z) &= \frac{z}{2}\left(\frac{k}{\gamma}\right)\left[\frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] - \left(\frac{1}{2}\right)^{\nu} z^{\nu-1}\nu W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right)\right] \\ &= \left(\frac{k}{\gamma}\right)\left[\frac{z}{2}\frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] - \left(\frac{z}{2}\right)^{\nu}\frac{\nu}{2}W_{k,\alpha,\nu+1}^{\gamma}\left(\frac{-z^2}{4}\right)\right] \\ &= \left(\frac{k}{\gamma}\right)\left[\frac{z}{2}\frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] - \frac{\nu}{2}J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] \end{aligned}$$

or equivalent

$$\left(\frac{\gamma}{k}\right) J_{k,\nu}^{(\gamma+k)(\alpha)}(z) - \left(\frac{2\gamma - k\nu}{2k}\right) J_{k,\nu}^{(\gamma)(\alpha)}(z) = \frac{z}{2}\frac{d}{dz}\left[J_{k,\nu}^{(\gamma)(\alpha)}(z)\right] \qquad \text{(II.10)}$$

Lemma 2 *Let* $k \in \mathbb{R}$*,* $\alpha, \gamma, \nu \in \mathbb{C}; Re(\alpha) > 0, Re(\nu + k) > 0$*. Then*

$$\left[(1+\nu) - \alpha\frac{(\nu+k)}{2}\right] J_{k,\nu+k}^{(\gamma)(\alpha)}(z) + \alpha\frac{z}{2}\frac{d}{dz}J_{k,\nu+k}^{(\gamma)(\alpha)}(z) = \left(\frac{z}{2}\right)^{k} J_{k,\nu}^{(\gamma)(\alpha)}(z) \qquad \text{(II.11)}$$

Proof.

Starting from the left hand member and using the Definition 6:

$$(1+\nu)J_{k,\nu+k}^{(\gamma)(\alpha)}(z) + \alpha\frac{z}{2}\frac{d}{dz}J_{k,\nu+k}^{(\gamma)(\alpha)}(z) - \alpha\frac{(\nu+k)}{2}\left(\frac{z}{2}\right)^{\nu+k} W_{k,\alpha,\nu+k+1}^{\gamma}\left(-\frac{z^2}{4}\right) =$$

$$(1+\nu)J_{k,\nu+k}^{(\gamma)(\alpha)}(z) + \alpha\frac{z}{2}\left[\frac{d}{dz}J_{k,\nu+k}^{(\gamma)(\alpha)}(z) - (\nu+k)\left(\frac{1}{2}\right)^{\nu+k} z^{\nu+k-1}W_{k,\alpha,\nu+k+1}^{\gamma}\left(-\frac{z^2}{4}\right)\right] = \qquad \text{(II.12)}$$

note that the expression of the bracket is equal to

$$\left(\frac{z}{2}\right)^{\nu+k}\frac{d}{dz}W_{k,\alpha,\nu+k+1}^{\gamma}\left(-\frac{z^2}{4}\right)$$

is calculated as

$$\frac{d}{dz}\left[\left(\frac{z}{2}\right)^{\nu+k} W_{k,\alpha,\nu+k+1}^{\gamma}\left(-\frac{z^2}{4}\right)\right]$$

back to (II.12)

$$= (1+\nu)J_{k,\nu+k}^{(\gamma)(\alpha)}(z) + \alpha\frac{z}{2}\left(\frac{z}{2}\right)^{\nu+k}\frac{d}{dz}W_{k,\alpha,\nu+k+1}^{\gamma}\left(-\frac{z^2}{4}\right)$$

$$= (1+\nu)\left(\frac{z}{2}\right)^{\nu+k} W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right) + \alpha\frac{z}{2}\left(\frac{z}{2}\right)^{\nu+k}\frac{d}{dz}W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right)$$
$$= (1+\nu)\left(\frac{z}{2}\right)^{\nu+k} W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right) + \alpha\frac{z}{2}\left(\frac{z}{2}\right)^{\nu+k}\frac{d}{dz}W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right)$$
$$= \left(\frac{z}{2}\right)^{\nu+k}\left[(1+\nu)W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right) + \alpha\frac{z}{2}\frac{d}{dz}W^{\gamma}_{k,\alpha,\nu+k+1}\left(-\frac{z^2}{4}\right)\right]$$

by Lemma 1 of [6],

$$= \left(\frac{z}{2}\right)^{\nu+k} W^{\gamma}_{k,\alpha,\nu+1}\left(-\frac{z^2}{4}\right)$$
$$= \left(\frac{z}{2}\right)^{k}\left(\frac{z}{2}\right)^{\nu} W^{\gamma}_{k,\alpha,\nu+1}\left(-\frac{z^2}{4}\right)$$
$$= \left(\frac{z}{2}\right)^{k} J^{(\gamma)(\alpha)}_{k,\nu}(z)$$

The following theorem relates the k-Mittag-Leffler function, k-Wright function and k-Bessel function of first kind.

Theorem 1 *If* $\nu, \gamma, \lambda \in \mathbb{C}$, *and* $w, k \in \mathbb{R}$. *Then*

$$\mathcal{L}\{(\sqrt{-\omega})^{-\nu} J^{(\gamma)(\lambda)}_{k,\nu}(\sqrt{-4\omega})\}(s) = \{W^{\gamma}_{k,\lambda,\nu+1}(\omega)\}(s) = s^{-1}E^{\gamma}_{k,\lambda,\nu+1}(s^{-}1) \qquad \text{(II.13)}$$

Proof.
Taking $z = \sqrt{-4\omega}$ $(\omega \in \mathbb{R})$ in the Definition 6, we have

$$J^{(\gamma)(\lambda)}_{k,\nu}(\sqrt{-4\omega}) = \left(\frac{\sqrt{-4\omega}}{2}\right)^{\nu} W^{\gamma}_{k,\lambda,\nu+1}\left(-\frac{(\sqrt{-4\omega})^2}{4}\right) \qquad \text{(II.14)}$$
$$= \left(\frac{\sqrt{-4\omega}}{2}\right)^{\nu} W^{\gamma}_{k,\lambda,\nu+1}(\omega) \qquad \text{(II.15)}$$
$$= (\sqrt{-\omega})^{\nu} W^{\gamma}_{k,\lambda,\nu+1}(\omega) \qquad \text{(II.16)}$$

that is

$$(\sqrt{-\omega})^{-\nu} J^{(\gamma)(\lambda)}_{k,\nu}(\sqrt{-4\omega}) = W^{\gamma}_{k,\lambda,\nu+1}(\omega) \qquad \text{(II.17)}$$

Taking the Laplace transform to both sides of (II.17) and from [6] result:

$$\mathcal{L}\{(\sqrt{-\omega})^{-\nu} J^{(\gamma)(\lambda)}_{k,\nu}(\sqrt{-4\omega})\}(s) = \{W^{\gamma}_{k,\lambda,\nu+1}(\omega)\}(s) = s^{-1}E^{\gamma}_{k,\lambda,\nu+1}(s^{-}1)$$

References

[1] J. Dettman, *Applied Complex Variables*. Dover Publications, INC. New York 1970.

[2] R. Diaz and E. Pariguan, *On hypergeometric functions and k-Pochammer symbol*, Divulgaciones Matematicas Vol.15 2 (2007)pp. 179-192 arXiv: math0405596v2.

[3] Dorrego, G.; Cerutti, R. *The k-Mittag-Leffler function.* Journal of Applied Math. Int. J. Contemp. Math. Sciences. Vol 7. N.15. 2012.

[4] Levedev, N. N. *Special Functions and their Applications.* Dover. 1972.

[5] Mainardi, F. *On the distinguished role of the Mittag-Leffler and Wright functions in fractional calculus.* Special Functions in the 21st Century: Theory and Applications. Washington DC, USA, 6-8 April 2011.

[6] Romero, L; Cerutti, R. *Fractional calculus of a k-Wright type function.* To appear.

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