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On the k-Bessel Functions

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Abstract

In this brief paper introduces some k-generalizations of the so-called special functions as Bessel functions and the Fox-Wright functions.

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I Introduction and Preliminaries

Since Diaz and Pariguan (cf.[2]) have introduced the k-gamma function $\Gamma_k(z)$ and the generalized Pochhammer k-symbol, several articles heve been devoted to studying generalizations of some of the so-called special functions. So can be found the k-Beta function, the k-Zeta function, the k-Mittag-Leffler function and the k-Wright function.

The integral expression of the k-gamma function is given by

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt,\ Re(z) > 0,\, k > 0. \tag{I.1}$$

Whose relationship with the classical Gamma Euler functions is given by

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right) \tag{I.2}$$

We collect some of its properties in the following

Lemma 1 *The k-gamma function $\Gamma_k(z)$ verified that:*

1. $\Gamma_k(z+k) = z\Gamma(z)$

2. $\Gamma_k(k) = 1$

3. Let $a \in \mathbb{R}$,

$$\Gamma_k(z) = a^{\frac{z}{k}} \int_0^\infty t^{z-1} e^{-\frac{z^k}{k}a} dt. \tag{I.3}$$

4. $\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{\sin(\pi z/k)}$

For the proof, that we omit, we refer to [2].

Right now we also have the k-Beta function $B_k(z)$ that is defined by the formula

$$B_k(z,w) = \frac{\Gamma_k(z)\Gamma_k(w)}{\Gamma_k(z+w)}; \quad Re(z) > 0,\ Re(w) > 0. \tag{I.4}$$

that have the integral representation given by

$$B_k(z,w) = \frac{1}{k} \int_0^\infty t^{\frac{z}{k}-1}(1-t)^{\frac{w}{k}-1} dt \tag{I.5}$$

Two functions widely used in fractional calculus because of the importance of their roles in the solution of fractional differential equations are the Mittag-Leffler function $E_\alpha(z)$ and the Wright functions $W(z)$.

The Mittag-Leffler function is an entire function defined by

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \tag{I.6}$$

A first generalization is given by a more general series

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \beta > 0. \tag{I.7}$$

called the Mittag-Leffler of two parameters.

From (I.6) and (I.7) we have

$$E_{\alpha,1}(z) = E_\alpha(z)$$

Another generalization was done by Prabhakar (cf.[7]) who introduced the Mittag-Leffler type function $E^\gamma_{\alpha,\beta}(z)$ defined by

$$E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad Re(\alpha) > 0,\ Re(\beta) > 0 \tag{I.8}$$

with $(\gamma)_n$ the Pochhammer symbol given by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)...(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \tag{I.9}$$

In a recent paper of us (cf.[3]) we have defined a new Mittag-Leffler type function as the series

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \tag{I.10}$$

valid for $Re(\alpha) > 0$, $Re(\beta) > 0$, $\gamma \in \mathbb{C}$ and $(\gamma)_{n,k}$ is the Pochhammer k-symbol given by

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+(n-1)k), \ \ \gamma \in \mathbb{C}, \ k \in \mathbb{R}, \ n \in \mathbb{N}. \tag{I.11}$$

It may be observed that $E^{\gamma}_{k,\alpha,\beta}(z)$ is such that $E^{\gamma}_{k,\alpha,\beta}(z) \longrightarrow E^{\gamma}_{\alpha,\beta}(z)$ as $k \to 1$, since $(\gamma)_{n,k} \longrightarrow (\gamma)_n$, $\Gamma_k(z) \longrightarrow \Gamma(z)$ and the convergence of the series in (I.10) is uniform on compact subsets.

Also we have defined (cf.[**?**]) the k-Wright type function as the series

$$W^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{(n!)^2} \tag{I.12}$$

for $Re(\alpha) > -1$, $Re(\beta) > 0$, $k \in \mathbb{R}$, $n \in \mathbb{N}$.

Can be easily seen that when $\gamma = 1$ and $k = 1$ (I.12) reduces to the classical Wright function

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{I.13}$$

II k-Bessel functions

Based on the well know relation (cf.[5])

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(-\frac{z^2}{4}\right) \tag{II.1}$$

where $W_{\lambda,\nu}(z)$ is the Wright function defined in (I.13) and $J_\nu(z)$ is the Bessel function of the first kind of order ν (cf.[**?**]) given by the series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(n+\nu+1)} \tag{II.2}$$

we have defined (cf.[1]) the k-Bessel function of the first kind $J_{k,\nu}^{(\gamma)(\lambda)}(z)$ as

$$J_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)} \frac{(-1)^n (z/2)^n}{(n!)^2} \tag{II.3}$$

where $(\gamma)_{n,k}$ is the Pochhammer k-symbol and $\Gamma_k(z)$ is the k-gamma function.

From (I.13) and (II.3) it may be write

$$J_{k,\nu}^{\gamma,\lambda}(z) = \left(\frac{z}{2}\right)^\nu W_{k,\lambda,\nu+1}^{\gamma}\left(-\frac{z^2}{4}\right). \tag{II.4}$$

Next, we put the following

Definition 1 *The k-modified Bessel function of the first kind of order ν (or $-\nu$ respectively) as*

$$I_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+k)} \frac{(z/2)^{\nu+2n}}{(n!)^2} \tag{II.5}$$

or

$$I_{k,-\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n-\nu+k)} \frac{(z/2)^{2n-\nu}}{(n!)^2} \tag{II.6}$$

In terms of the k-Wright function we have

$$I_{k,\nu}^{\gamma,\lambda}(z) = \left(\frac{z}{2}\right)^\nu W_{k,\lambda,\nu+k}^{\gamma}\left(\frac{z^2}{4}\right). \tag{II.7}$$

Also we have following

Definition 2 *The k-modified Bessel function of the third kind $K_{k,\nu}^{\gamma,\lambda}(z)$ is*

$$K_{k,\nu}^{\gamma,\lambda}(z) = \frac{\pi\left[I_{k,-\nu}^{\gamma,\lambda}(z) - I_{k,\nu}^{\gamma,\lambda}(z)\right]}{2\sin(\nu\pi)} \tag{II.8}$$

Now, we will show some elementary properties.

Lemma 2 *Let ν be a complex number, $Re(\nu) > 0$ and let k, γ, z be real non negative numbers. For $\lambda = 1$ holds*

$$\frac{d}{dz}\left(z^{\nu/2} I^{\gamma,1}_{k,\nu}(\sqrt{z})\right) = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I^{\gamma,1}_{k,\nu-k}(\sqrt{z}) \tag{II.9}$$

Proof. From Definition (II.5) we have

$$z^{\nu/2} I^{\gamma,1}_{k,\nu}(\sqrt{z}) = z^{\nu/2} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\sqrt{z}/2)^{\nu+2n}}{\Gamma_k(n+\nu+k)(n!)^2} =$$

$$= \frac{1}{2^\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{(n!)^2 4^n} \frac{z^{\nu+n}}{\Gamma_k(n+\nu+k)} \tag{II.10}$$

Then

$$\begin{aligned}
\frac{d}{dz}\left(z^{\nu/2} I^{\gamma,1}_{k,\nu}(\sqrt{z})\right) &= \frac{1}{2^\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{(n!)^2 4^n} \frac{(\nu+n) z^{\nu+n-1}}{\Gamma_k(n+\nu+k)} \\
&= 2^{-k} z^{\frac{\nu-1}{2}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\sqrt{z}/2)^{\nu-1+2n}}{(n!)^2 \Gamma_k(\nu+n)} \\
&= 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\sqrt{z}/2)^{(\nu-k)+2n}}{(n!)^2 \Gamma_k(\nu+n)} \\
&= 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I^{\gamma,1}_{k,\nu-k}(\sqrt{z}).
\end{aligned}$$

Lemma 3 *Let $I^{\gamma,1}_{k,-\nu}(z)$ be the k-modified Bessel function of the first kind of order $-\nu$, and let z be a real non negative number. Then holds*

$$\frac{d}{dz}\left(z^{\nu/2} I^{\gamma,1}_{k,-\nu}(\sqrt{z})\right) = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I^{\gamma,1}_{k,-\nu-k}(\sqrt{z}) \tag{II.11}$$

The proof is completely analogous to the Lemma 2 and then we omit it.

Lemma 4 *Let ν be a complex number, $Re(\nu) > 0$ and let k, γ, z be real non negative numbers, and $\lambda = 1$. Then:*

$$\frac{d}{dz}\left[z^{z/2} K^{\gamma,1}_{k,\nu}(\sqrt{z})\right] = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} K^{\gamma,1}_{k,\nu-k}(\sqrt{z}) \tag{II.12}$$

Proof. From (II.8), (II.9) and (II.10) we have

$$z^{\nu/2}K^{\gamma,1}_{k,\nu}(\sqrt{z}) = z^{\nu/2}\left[I^{\gamma,1}_{k,-\nu}(\sqrt{z}) - I^{\gamma,1}_{k,\nu}(\sqrt{z})\right]\frac{\pi}{2\sin(\nu\pi)}$$

Thus

$$\begin{aligned}\frac{d}{dz}\left[z^{\nu/2}K^{\gamma,1}_{k,\nu}(\sqrt{z})\right] &= \frac{2^{-k}z^{\frac{\nu-k}{2}}z^{k-1}\pi}{2\sin(\nu\pi)}\left[I^{\gamma,1}_{k,-\nu-1}(\sqrt{z}) - I^{\gamma,1}_{k,\nu+1}(\sqrt{z})\right]\\ &= \frac{2^{-k}z^{\frac{\nu-k}{2}}z^{k-1}2\sin(\nu+1)\pi}{2\sin(\nu\pi)}\left[\frac{I^{\gamma,1}_{k,-\nu-1}(\sqrt{z}) - I^{\gamma,1}_{k,\nu+1}(\sqrt{z})}{2\sin(\nu+1)\pi}\right]\\ &= -2^{-k}z^{\frac{\nu-k}{2}}z^{k-1}K^{\gamma,1}_{k,\nu+1}(\sqrt{z}).\end{aligned}$$

In the next we will use a fractional integral called k-fractional integral (cf.[6]) that is a k-generalization of the classical Riemann-Liouville fractional integral (cf.[4]).

The k-fractional integral of order α is defined by

$$I^{\alpha}_{k}(f)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt \qquad \text{(II.13)}$$

It may be observed that when $k \to 1$, (II.13) reduces to the classical Riemann-Liouville fractional integral.

Taking into account that (cf.[6])

$$I^{\alpha}_{k}\left(\frac{x^{\frac{\beta}{k}-1}}{\Gamma_k(\beta)}\right) = \frac{x^{\frac{\alpha}{k}+\frac{\beta}{k}+1}}{\Gamma_k(\alpha+\beta)}, \qquad \text{(II.14)}$$

we have the following

Lemma 5

$$I^{\alpha}_{k}\left(z^{\nu/2}I^{\gamma,1}_{k,\nu}(2\sqrt{z})\right) = z^{\frac{\alpha}{k}+\nu}\ {}_{k,1}\Psi_2\left[\begin{array}{c}(k(\nu+1),k)\\(\alpha+k(\nu+1),k),((\nu+k),1)\end{array}\ \Big|\ z\right] \qquad \text{(II.15)}$$

Proof. By the uniform convergence on compact subsets of the series $I^{\gamma,1}_{k,\nu}(z)$, from (II.10) and (II.14) we have

$$\begin{aligned}I^{\alpha}_{k}\left(z^{\nu/2}I^{\gamma,1}_{k,\nu}(2\sqrt{z})\right) &= \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{(n!)^2}\frac{\Gamma_k(k(\nu+1)+kn)}{\Gamma_k(k(\nu+1)+\alpha+kn)}\frac{z^{\frac{\alpha}{k}+\nu+n}}{\Gamma_k(n+\nu+k)}\\ &= z^{\frac{\alpha}{k}+\nu}\ {}_{k,1}\Psi_2\left[\begin{array}{c}(k(\nu+1),k)\\(\alpha+k(\nu+1),k),((\nu+k),1)\end{array}\ \Big|\ z\right]\end{aligned}$$

where ${}_{k,1}\Psi_2$ denote the k-Fox-Wright function.

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