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Transitions among $SO(3)/\sim$, $S^{2+}/\sim$ and $Q_0/\sim$

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Abstract

In this paper we construct transitions among $SO(3)/\sim$, $S^{2+}/\sim$ and $Q_0/\sim$ using equivalence relations on $SO(3)$ and S^{2+}, where $S^{2+} = S^2 \cup \{(0,0,0)\}$, S^2 is a unit sphere, Q_0 is the set of unit quaternions and $SO(3)$ is the set of special orthogonal matrix of 3×3. Firstly we construct equivalence relations on $SO(3)$, S^2 and Q_0. Finally we introduce the transitions among $SO(3)/\sim$, $S^{2+}/\sim$ and $Q_0/\sim$ and give Matlab applications.

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1 Introduction

If a mapping f from $\mathbb{R}^3$ to $\mathbb{R}^3$ has the following two corollaries, it is called orthogonal mapping.
1. $f(0) = 0$ and
2. $d(f(x), f(y)) = d(x, y)$, $\forall x, y \in \mathbb{R}^3$
A matrix form of an orthogonal mapping is orthogonal matrix [4,6]. The set special orthogonal matrix

$$SO(3) = \{A \in \mathbb{R}^3_3 \mid AA^T = I, \det A = 1\}$$

is a group under matrix multiplication and is called special orthogonal group. Each special orthogonal matrix defines a rotation around origin. Characteristic vector of a special orthogonal matrix for $\lambda = 1$ characteristic value defines a line passing origin which is rotating axis of rotation around origin. Each $A \in SO(3)$, we know what direction line and rotation angle are. Direction line has direction vector as characteristic vector for $\lambda = 1$ and rotation angle is

$$\theta = arccos\frac{tr(A)-1}{2}. \tag{1}$$

In $\mathbb{R}^3$, a rotation around an axis passing origin in the plane is perpendicular to this axis. The axis is called rotating axis and perpendicular plane which called rotating plane. The same rotation can be given by using unit quaternion and quaternion product [1,2,3]. A unit quaternion has the form

$$q = cos(\theta) + \overrightarrow{\varepsilon_q}.sin(\theta)$$

$\overrightarrow{\varepsilon_q}$ is a unit vector and defines rotation axis. $\overrightarrow{\varepsilon_q}$ and unit vector of characteristic vector define a point in S^2. Hence, a relation can be established among S^2, Q_0 and $SO(3)$. For $S^{2+} = S^2 \cup \{(0,0,0)\}$, the relation between $SO(3)/\sim$ and $S^{2+}/\sim$ is given [7]. This relation summarizes in Section 2. Same relation can be established between $Q_0/\sim$ and $S^{2+}/\sim$. So a transition can be given among $SO(3)/\sim$, $Q_0/\sim$ and $S^{2+}/\sim$. In this article we study these transitions and give some Matlab applications.

2 REPRESENTATION OF $SO(3)/\sim$ AT $S^{2+}/\sim$

A group structure on $S^{2+}/\sim$, $S^{2+} = S^2 \cup \{(0,0,0)\}$ is constructed. Thus an equivalence relation $\sim$ on $SO(3)$ and one-to-one correspondence between $S^{2+}/\sim$ and $SO(3)/\sim$ are used [7].

A rotating matrix around an axis $\overrightarrow{b}$ is known with components of $\overrightarrow{b}$.

Rotation matrix about an arbitrary axis with θ rotating angle

$$R_\theta = \begin{bmatrix} b_1^2(1-\cos\theta)+\cos\theta & b_1b_2(1-\cos\theta)-b_3\sin\theta & b_1b_3(1-\cos\theta)+b_2\sin\theta \\ b_1b_2(1-\cos\theta)+b_3\sin\theta & b_2^2(1-\cos\theta)+\cos\theta & b_2b_3(1-\cos\theta)-b_1\sin\theta \\ b_1b_3(1-\cos\theta)-b_2\sin\theta & b_2b_3(1-\cos\theta)+b_1\sin\theta & b_3^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{2}$$

where $\vec{b} = (b_1, b_2, b_3)$ and $\left\|\vec{b}\right\| = 1$ [5].

On the other hand, having the same rotating axis doesn't require to have the same rotating angle. A rotation in $\mathbb{R}^3$ is characterized with rotation axis and rotating angle. All rotations with the same rotating axis can be taken as an equivalence class. For this we define a relation on $SO(3)$. So we have,

Theorem 2.1 *Let $\sim$ relation defines as*

$$A \sim B \Leftrightarrow \overrightarrow{X_A} \parallel \overrightarrow{X_B}$$

on $SO(3)$; where, $\overrightarrow{X_A}$ and $\overrightarrow{X_B}$ are characteristic vectors of A and B associated with $\lambda_1 = 1$,respectively. $\sim$ is an equivalence relation on $SO(3)$

[7].

$$SO(3)/\sim = \{[A] \mid A \in SO(3)\}$$

The representing element of $[A] \in SO(3)/\sim$ can be taken a matrix which its characteristic vector is unit.

Theorem 2.2 *Let $\diamond$ operation defines as*

$$[A] \diamond [B] = [A.B]$$

on $SO(3)/\sim$, where . is matrix product. So $(SO(3)/\sim, \diamond)$ is a group

[7].

Theorem 2.3 $(S^{2+}/\sim, *)$ *is a group*

[7].

Lemma 2.4 *A mapping*

$$F : SO(3)/\sim \longrightarrow S^{2+}/\sim$$

$$F([A]) = \begin{cases} X_A, [A] \neq [I] \\ 0, [A] = [I] \end{cases}$$

is one-to-one and onto.
F is one-to-one:
For all $[A], [B] \in SO(3)/\sim$,

$$F([A]) = F([B]) \Rightarrow X_A = X_B \Rightarrow \overrightarrow{X_A} = \overrightarrow{X_B} \Rightarrow \overrightarrow{X_A} \parallel \overrightarrow{X_B} \Rightarrow [A] = [B]$$

F is onto:

$$P \in S^{2+}/\sim \Rightarrow \left\| \overrightarrow{OP} \right\| = 1 or P = 0$$

is known.

$$P = 0 \Rightarrow F^{-1}(0) = [I].$$

$$P \neq 0 \Rightarrow$$

an A orthogonal matrix which has $\overrightarrow{OP}$ as a characteristic vector can be obtained using Cayley's formula.

3 GROUP STRUCTURE ON $Q_0/\sim$

In 1843 Hamilton invented the so-called hyper-complex number of rank 4, to which he gave the name quaternion. Now a quaternion, as the name already suggests, may be regarded as a 4-tuple of real numbers, that is, as an element of $\mathbb{R}^4$. In this case we would write

$$\mathbf{q} = (q_0, q_1, q_2, q_3)$$

where q_0, q_1, q_2 and q_3 are simply real numbers or scalars. We define a quaternion as the sum

$$\mathbf{q} = q_0 + \overrightarrow{q} = q_0 + \overrightarrow{i} q_1 + \overrightarrow{j} q_2 + \overrightarrow{k} q_3.$$

In this sum, q_0 is called the scalar part of the quaternion while $\overrightarrow{q}$ is called the vector part of the quaternion. The scalars q_0, q_1, q_2, q_3 are called the components of the quaternion.

The sum of the two quaternions p and q is defined by adding the corresponding components, that is

$$\mathbf{p} + \mathbf{q} = (p_0 + q_0) + \overrightarrow{i}(p_1 + q_1) + \overrightarrow{j}(p_2 + q_2) + \overrightarrow{k}(p_3 + q_3).$$

Each quaternion $\mathbf{q}$ has a negative or an additive inverse, denoted $-\mathbf{q}$ in which each component is the negative of the corresponding component of $\mathbf{q}$.

We may write the product of the two quaternions $\mathbf{p} = p_0 + \overrightarrow{p}$ and $\mathbf{q} = q_0 + \overrightarrow{q}$ in the more concise form

$$\mathbf{p} \circledast \mathbf{q} = p_0 q_0 + p_0 \overrightarrow{q} + q_0 \overrightarrow{p} - \langle \overrightarrow{p}, \overrightarrow{q} \rangle + \overrightarrow{p} \times \overrightarrow{q}.$$

The complex conjugate of the quaternion

$$\mathbf{q} = q_0 + \overrightarrow{q} = q_0 + \overrightarrow{i} q_1 + \overrightarrow{j} q_2 + \overrightarrow{k} q_3$$

is given by

$$\mathbf{q}^* = q_0 - \overrightarrow{q} = q_0 - \overrightarrow{i} q_1 - \overrightarrow{j} q_2 - \overrightarrow{k} q_3.$$

The norm of a quaternion q, denoted by $N(\mathbf{q})$ or $|\mathbf{q}|$, is the scalar defined by

$$N(\mathbf{q}) = \sqrt{\mathbf{q}^* \mathbf{q}}.$$

Using the ideas of the complex conjugate and the norm of a quaternion, we are now able to show that every non-zero quaternion does have a multiplicative inverse, and we can develop a formula for it. If we designate the inverse $\mathbf{q}^{-1}$ we must, by definition of inverse, have

$$\mathbf{q}^{-1}\mathbf{q} = \mathbf{q}\mathbf{q}^{-1} = 1.$$

Now, if we use both pre- and post- multiplication by the complex conjugate q^* we may write

$$\mathbf{q}^{-1}\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q}\mathbf{q}^{-1} = \mathbf{q}^*.$$

Since $\mathbf{q}\mathbf{q}^* = N^2(\mathbf{q})$ we get

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{N^2(\mathbf{q})} = \frac{\mathbf{q}^*}{|\mathbf{q}|^2}.$$

We note here that if q is a unit or normalized quaternion, that is, $N(\mathbf{q}) = 1$, then the inverse is simply the complex conjugate

$$\mathbf{q}^{-1} = \mathbf{q}^*.$$

It is possible to explain the rotating motion, R_θ, with quaternion and quaternion product. A unit quaternion

$$\mathbf{q} = d + a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$$

can be written as follows:

$$\mathbf{q} = d + \frac{a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}}{\sqrt{a^2+b^2+c^2}}\sqrt{a^2+b^2+c^2}$$

or

$$\mathbf{q} = \cos\theta + \overrightarrow{\varepsilon_{\mathbf{q}}}\sin\theta.$$

A mapping

$$L_{\mathbf{q}}(v) = \mathbf{q} \circledast v \circledast \mathbf{q}^*$$

defines a rotating about $\overrightarrow{\varepsilon_{\mathbf{q}}}$ with 2θ angle. The matrix form of $L_{\mathbf{q}}$ is

$$L_{\mathbf{q}} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix}$$

[2,3]. To have the same rotating axis don't require to have the same rotating angle. The distinctive property between unit quaternions, which has the same axis on

$$Q_0 = \{\mathbf{q} \in Q \mid \|\mathbf{q}\| = 1\},$$

is $d = \cos\theta$. The same axis and the different d values cause an equivalence class.

So we have

Theorem 3.1 *Let we define a relation $\sim$ on Q_0 as*

$$\mathbf{p} \sim \mathbf{q} \Leftrightarrow \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{q}}}.$$

$\sim$ is an equivalence relation on Q_0.
For $\mathbf{p}, \mathbf{q} \in Q_0$, $\mathbf{p} \sim \mathbf{q} \Leftrightarrow \mathbf{p}$ and $\mathbf{q}$ have the same rotating axis ($\overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{q}}}$).
i) $\mathbf{p} \sim \mathbf{p} \Leftrightarrow \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{p}}}$ for all $\mathbf{p} \in Q_0$
ii) $\mathbf{p} \sim \mathbf{q} \Rightarrow \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{q}}} \Rightarrow \overrightarrow{\varepsilon_{\mathbf{q}}} \parallel \overrightarrow{\varepsilon_{\mathbf{p}}} \Rightarrow \mathbf{q} \sim \mathbf{p}$
iii) $\mathbf{p} \sim \mathbf{q}$ and $\mathbf{q} \sim \mathbf{r} \Rightarrow \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{q}}}$ and $\overrightarrow{\varepsilon_{\mathbf{q}}} \parallel \overrightarrow{\varepsilon_{\mathbf{r}}} \Rightarrow \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{r}}} \Rightarrow \mathbf{p} \sim \mathbf{r}$.

So we have an equivalence class on $Q_0/\sim$ as, for $\mathbf{q} \in Q_0$, $[\mathbf{q}] \in Q_0/\sim$ and

$$[\mathbf{q}] = \{\mathbf{p} \in Q_0 \mid \overrightarrow{\varepsilon_{\mathbf{p}}} \parallel \overrightarrow{\varepsilon_{\mathbf{q}}}\}.$$

There is a one-to-one corresponding between the set of quaternions ,

$$Q = \{\mathbf{q} \mid \mathbf{q} = d + a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}\}$$

and $\mathbb{R}^4$. There is a one-to-one corresponding unit quaternions,

$$Q_0 = \{\mathbf{q} \in Q \mid \|\mathbf{q}\| = 1\}$$

and unit hypersphere S^3. Now we will show that one-to-one corresponding between $S^{2+}/\sim$ and $Q_0/\sim$ can be establishment using equivalence class we give in Theorem 5.

Theorem 3.2 *$(Q_0/\sim, \boxdot)$ is a group.*
For $\forall\ [\mathbf{p}], [\mathbf{q}] \in Q_0/\sim$, $\mathbf{p}, \mathbf{q} \in Q_0 \Rightarrow \mathbf{p} \circledast \mathbf{q} \in Q_0 \Rightarrow [\mathbf{p}] \boxdot [\mathbf{q}] \in Q_0/\sim$.
Unit element: $e = (1, 0, 0, 0) \in Q_0$ is unit element. Because of this $[\mathbf{e}] \in Q_0/\sim$.
Inverse element: For $\forall\ [\mathbf{p}] \in Q_0/\sim \Rightarrow \mathbf{p} \in Q_0 \Rightarrow \mathbf{p}^{-1} \in Q_0 \Rightarrow [\mathbf{p}]^{-1} \in Q_0/\sim$.
Association property is obvious.

4 REPRESENTATION OF $Q_0/\sim$ AT $S^{2+}/\sim$

Theorem 4.1 *A mapping*

$$G : Q_0/\sim \rightarrow S^{2+}/\sim$$

$$G([\mathbf{q}]) = \begin{cases} P, d \neq \pm 1, \overrightarrow{\varepsilon_{\mathbf{q}}} = \overrightarrow{OP} \\ 0, d = \pm 1 \end{cases}$$

is one-to-one and onto.

G is one-to-one:
For all $[\mathbf{p}],[\mathbf{q}]\in Q_0/\sim$,

$$\begin{aligned} G([\mathbf{p}]) &= G([\mathbf{q}]) \Rightarrow \mathbf{p}=\mathbf{q} \Rightarrow \overrightarrow{OP}=\overrightarrow{OQ} \\ &\Rightarrow \overrightarrow{\varepsilon_{\mathbf{p}}}=\overrightarrow{\varepsilon_{\mathbf{q}}} \\ &\Rightarrow [\mathbf{p}]=[\mathbf{q}]. \end{aligned}$$

G is onto:

$$\begin{aligned} P &\in S^{2+}/\sim \Rightarrow \left\|\overrightarrow{OP}\right\|=1 \\ \Rightarrow For\overrightarrow{OP}=\overrightarrow{\varepsilon_{\mathbf{p}}},\cos\theta+\overrightarrow{\varepsilon_{\mathbf{q}}}\sin\theta & \\ isknownforany\theta, consequently[\mathbf{p}] &\in Q_0 isobtainedunique. \\ & or \\ P &= 0 \Rightarrow G^{-1}(0)=[\mathbf{e}]fromdefinitionofG. \end{aligned}$$

For $\forall\ [\mathbf{p}],[\mathbf{q}]\in Q_0/\sim$, let a product $\circledast$ defined as $[\mathbf{p}]\boxdot[\mathbf{q}]=[\mathbf{p}\circledast\mathbf{q}]$; where $\circledast$ is quaternion product.

We define an operation $\circledcirc$ on S^{2+}, as

$$\circledcirc : S^{2+}/\sim \times S^{2+}/\sim \longrightarrow S^{2+}/\sim$$

$$(P,Q)\rightarrow P\circledcirc Q=R,$$

where,

$$\begin{aligned} G^{-1}(P) &= [\mathbf{p}], \\ G^{-1}(Q) &= [\mathbf{q}] \end{aligned}$$

and

$$\begin{aligned} P\circledast Q &= R, \\ R &= G([\mathbf{r}]). \end{aligned}$$

5 TRANSITIONS AMONG $SO(3)/\sim$, $S^{2+}/\sim$ AND $Q_0/\sim$

We give the group structure on $S^{2+}/\sim$ in [4]. One-to-one corresponding between $S^{2+}/\sim$, $SO(3)/\sim$ and $Q_0/\sim$ are used for these group structures. Also the group structures are used on $SO(3)/\sim$ and $Q_0/\sim$.

$A \in SO(3)$ and $[A] \in SO(3)/\sim$ can be defined for any $\overrightarrow{b} \in \mathbb{R}^3$, $\left\| \overrightarrow{b} \right\| = 1$.

For every $A \in SO(3)$, $\overrightarrow{x_A}$ characteristic vector for $\lambda = 1$ and $\theta = \arccos \frac{trA-1}{2}$, rotation angle, are known. A unit quaternion which has the same rotation axis and the same rotation angle with these informations. Namely, for $x_A = (a, b, c)$ and $\overline{d} = \cot \frac{\theta}{2}$,

$$\mathbf{q} = d + a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$$

is a quaternion which gives the rotation operator which has the same rotation axis and the same rotation angle; where,

$$d = \frac{\overline{d}}{\sqrt{\overline{d}^2 + 1}}, a_i = \frac{b_i}{\sqrt{\overline{d}^2 + 1}}, i = 1, 2, 3.$$

In this case, a one-to-one corresponding is constructed between $SO(3)/\sim$ and $Q_0/\sim$.

6 APPLICATIONS AT MATLAB

When a unit directing vector of a line is given, the rotating matrix which accepts this line as the rotating axis, quaternion form of quaternion operator which accepts the same axis as the rotating axis and calculation of matrix form of quaternion operator at Matlab programme are as follows:

```
clear all, close all, clc
                        fprintf('We chose A,B and C arbitrarily')
A=1;
B=2;
C=3;
v=[A B C]
N=(A^2+B^2+C^2)^(1/2);
a=A/N;
b=B/N;
c=C/N;
                        fprintf('a, b, c are component of rotation axis')
s=[a b c]
Q=30;
R=[a*a*(1-cosd(Q))+cosd(Q) a*b*(1-cosd(Q))-c*sind(Q) a*c*(1-cosd(Q))+b*sind(Q)
    a*b*(1-cosd(Q))+c*sind(Q) b*b*(1-cosd(Q))+cosd(Q) b*c*(1-cosd(Q))-a*sind(Q)
    a*c*(1-cosd(Q))-b*sind(Q) b*c*(1-cosd(Q))+a*sind(Q) c*c*(1-cosd(Q))+cosd(Q)]
tr=trace(R);
p=acosd((tr-1)/2);
d=cotd(p/2);
n=(d^2+a^2+b^2+c^2)^(1/2);
QQ=[cosd(p/2);a*sind(p/2);b*sind(p/2);c*sind(p/2)]
%QQ is a unit quaternion
 Q=[2*(d/n)^2-1+2*(a/n)^2   2*(a/n)*(b/n)-2*(d/n)*(c/n)  2*(a/n)*(c/n)+2*(d/n)*(b/n)
 2*(a/n)*(b/n)+2*(d/n)*(c/n)    2*(d/n)^2-1+2*(b/n)^2  2*(b/n)*(c/n)-2*(d/n)*(a/n)
 2*(a/n)*(c/n)-2*(b/n)*(d/n)  2*(b/n)*(c/n)+2*(d/n)*(a/n)  2*(d/n)^2-1+2*(c/n)^2]
                           fprintf('Q is a rotating matrix from quaternion Q')
F=[1;3;5];
                           fprintf('F is arbitrary point in R^3')
[V,D]= eig(R)
[V,D]= eig(Q)
V=V(:,1)
                           fprintf('we have same point under R or Q as ')
r=R*F
q=Q*F
% f,g,Q*Q' and R*R'are control values
f= det(Q)
g = det(R)
I1=Q*Q'
I2=R*R'
```

Running programme, if we choose a point in R^3 as $(1, 2, 3)$ so we have following values

```
We chose A,B and C arbitrarily
v =  1     2     3

a b c are component of rotation axis
s = 0.2673    0.5345    0.8018


R =
    0.8756   -0.3818    0.2960
    0.4200    0.9043   -0.0762
   -0.2386    0.1910    0.9522


QQ =
    0.9659
    0.0692
    0.1383
    0.2075


Q = 0.8756   -0.3818    0.2960
    0.4200    0.9043   -0.0762
   -0.2386    0.1910    0.9522

Q is a rotating matrix from quaternion QF is arbitrary point in R^3
V =
   0.6814             0.6814             0.2673
  -0.1048 - 0.5883i  -0.1048 + 0.5883i   0.5345
  -0.1572 + 0.3922i  -0.1572 - 0.3922i   0.8018


D =
   0.8660 + 0.5000i        0                  0
        0             0.8660 - 0.5000i        0
        0                  0             1.0000



D =
   0.8660 + 0.5000i        0                  0
        0             0.8660 - 0.5000i        0
        0                  0             1.0000


V =
   0.6814
  -0.1048 - 0.5883i
  -0.1572 + 0.3922i

we have same point under R or Q as
r =
    1.2102
    2.7519
    5.0954


q =
    1.2102
    2.7519
    5.0954

f =1
g =1

I1 =
    1.0000    0.0000         0
    0.0000    1.0000    0.0000
         0    0.0000    1.0000

I2 =
    1.0000         0         0
         0    1.0000         0
         0         0    1.0000
```

7 Conclusion

$SO(3)/\sim$ and $Q_0/\sim$ have group structure. Every equivalence class of $SO(3)/\sim$ and $Q_0/\sim$ corresponds to a point on $S^{2+}/\sim$. So we can construct two group structures on $S^{2+}/\sim$ using one-to-one corresponding between $S^{2+}/\sim$ and $SO(3)/\sim$. Also we construct a group structure on $S^{2+}/\sim$ using one-to-one corresponding between $S^{2+}/\sim$ and $Q_0/\sim$. Using two group structures on $S^{2+}/\sim$ and one-to-one corresponding between $SO(3)/\sim$ and $Q_0/\sim$, we can establish a translation among $S^{2+}/\sim$, $SO(3)/\sim$ and $Q_0/\sim$; where, $S^{2+}/\sim$ has a central role. When we take a point $p \in S^{2+}/\sim$, we can find an orthogonal matrix A and a unit quaternion $\mathbf{q}$. A and $\mathbf{q}$ define same rotation on $\mathbb{R}^3$. In addition, we give some applications on Matlab.

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