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Matlab Applications for Cone Surfaces of 1-Parameter Motions

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Abstract

This paper defines a regular curve in $\mathbb{R}^3$ by using characteristic vectors of a regular curve in $SO(3)$. This curve is called characteristic curve of 1-parameter motion. In this article we define as well the cone surfaces of a characteristic curve and obtain an algorithm for drawing cone surface whose base curve is a characteristic curve. Moreover, we provide several Matlab applications.

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1 Introduction

A displacement has two parts. They are rotations and translations. A rotation is an orientation-preserving orthogonal transformation. Euler's rotation theorem states that an arbitrary rotation can be parameterized using three parameters. These parameters are commonly taken as the Euler angles. Rotations can be implemented using rotation matrices. From Euler's rotation theorem we know that any rotation can be expressed as a single rotation about some axis. The axis remains unchanged by the rotation. A cone surface can be defined using all of these axis [2],[3],[6].

A mapping between two metric spaces which preserve the distance is called an isometry. If $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ and metric is Euclidean metric and $d(\varphi(x), \varphi(y)) =$

$d(x, y)$ then φ is an isometry on $\mathbb{R}^3$. If φ is an isometry and $\varphi(P) = P$ then φ is called a rotation around point P.

If $\overrightarrow{b} = (b_1, b_2, b_3)$ is a vector then

$$B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

is an skew-symmetric matrix. Depending on this, an orthogonal matrix

$$A = (I - B)^{-1}(I + B)$$

is obtained and $A\overrightarrow{b} = \overrightarrow{b}$.
The set of all special orthogonal matrices $SO(3)$ is a differential manifold and Lie group [1],[4],[7].

The set of points in the moving body, which do not change position under the action of A, is a solutions to the matrix equation

$$Ax = x.$$

We generalize this to the matrix eigenvalue problem

$$Ax = \lambda x. \tag{1}$$

This indicates that the coordinates X in F are a constant λ times the original coordinates x in M.

For (2.1), in order to find solutions other than $x = 0$, the determinant of $A - \lambda I$ must be zero. For a general 3×3 matrix this determinant yields the characteristic polynomial:

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(M_{11} + M_{22} + M_{33}) + \det A = 0,$$

where M_{ii} is the minor obtained by eliminating row i and column i. For a rotation matrix, we have $detA = 1$ and $M_{ii} = a_{ii}$, so this equation becomes

$$\lambda^3 - \lambda^2(a_{11} + a_{22} + a_{33}) + \lambda(a_{11} + a_{22} + a_{33}) - 1 = 0.$$

We see immediately that $\lambda = 1$ is a root, so the characteristic polynomial can be factored to obtain

$$(\lambda - 1)(\lambda^2 - \lambda(a_{11} + a_{22} + a_{33} - 1) + 1) = 0.$$

The remaining roots are $\lambda = \exp(i\phi)$ and $\lambda = \exp(-i\phi)$, where ϕ is defined by

$$\cos\phi = \frac{(a_{11} + a_{22} + a_{33} - 1)}{2}.$$

Let $\mathbf{b}$ be the eigenvector of A associated with $\lambda = 1$, all points on the line $\mathbf{v} = t\mathbf{b}$ in the direction $\mathbf{b}$ are fixed during the rotation. This is the axis of rotation of the body.

The complex eigenvectors $\mathbf{x}$ and $\mathbf{x}^*$ associated with $\lambda = \exp(i\phi)$ and $\lambda = \exp(-i\phi)$ define a plane perpendicular to $\mathbf{b}$. [3], [5].

2 Eigencurve and Cone Surface of an Orthogonal Curve

In this section we define a regular curve in $\mathbb{R}^3$ from the curve in $SO(3)$. Let $\alpha(t)$ is a curve in $SO(3)$. As we know $SO(3)$ is a 3-dimensional differentiable manifold. Every point of $\alpha(t)$ is an orthogonal matrix. The eigenvector of $\alpha(t)$, for eigenvalue 1, defines a line passing origin which is rotating axis and rotating angle is $\theta = arccos(\frac{1}{2}(tr(\alpha(t)) - 1))$

If $\overrightarrow{X(t)} = (x(t), y(t), z(t))$ is eigenvector of $\alpha(t)$, then $C(t) = (x(t), y(t), z(t))$ is a curve in $\mathbb{R}^3$, and we have

$$x(t) = \frac{(\alpha_{22}(t) - 1)(\alpha_{33}(t) - 1) - \alpha_{32}(t)\alpha_{23}(t)}{\alpha_{21}(t)\alpha_{32}(t) - \alpha_{31}(t)(\alpha_{22}(t) - 1)} c \tag{2}$$

$$y(t) = \frac{\alpha_{21}(t)(\alpha_{33}(t) - 1) - \alpha_{31}(t)\alpha_{23}(t)}{\alpha_{31}(t)(\alpha_{22}(t) - 1) - \alpha_{21}(t)\alpha_{32}(t)} c \tag{3}$$

$$z(t) = c, (c \in \mathbb{R}). \tag{4}$$

So we can define eigencurve as follows.

Definition 2.1 *A $C(t)$ curve in $\mathbb{R}^3$ defined as*

$$C : I \to \mathbb{R}^3, t \to C(t),$$

is called eigencurve *(*characteristic curve*) of $\alpha(t)$ orthogonal curve.*

The $C(t)$ curve is defined using characteristic vector of $\alpha(t)$. Every vector $X(t)$ has the property

$$\alpha(t)(X(t)) = X(t), \parallel X(t) \parallel = r_t, r_t \neq 0$$

. We see from (2.1-2.2) that, if $a_{13} \neq a_{31}$ then $C(t)$ is a regular curve.

If $\alpha : I \subseteq \mathbb{R} \to SO(3)$ an orthogonal curve, then $\alpha(t)$ is an orthogonal matrix, for every t. For every $\alpha(t)$ has a characteristic vector for $\lambda = 1$ eigenvalue. The eigenvector corresponding to eigenvalue $\lambda = 1$ defines a straight line passing through origin. The set of all these lines defines a cone surface in $\mathbb{R}^3$. This surface is called **cone surface of the orthogonal curve** $\alpha(t)$.

There are a lot of methods for writing an orthogonal matrix and construction an orthogonal curve. One of them is by using Euler's method. Nowadays, the angles in this method are called Euler parameters . The other method is by using unit vectors and Cayley's theorem, etc. Furthermore we can give some examples to the forms of orthogonal matrices.

For example;

Example 2.2 *a) For every $t \in R$ $A(t)$ is an orthogonal matrix*

$$A(t) = \frac{1}{1+t+t^2} \begin{pmatrix} -t & t+t^2 & 1+t \\ 1+t & -t & t+t^2 \\ t+t^2 & 1+t & -t \end{pmatrix}. \tag{5}$$

b) If we take a matrix as

$$B = \begin{pmatrix} a^2(1-d)+d & ab(1-d)-ce & ac(1-d)+be \\ ab(1-d)+ce & b^2(1-d)+d & bc(1-d)-ae \\ ac(1-d)-be & bc(1-d)+ae & c^2(1-d)+d \end{pmatrix} \tag{6}$$

where $a^2+b^2+c^2 = 1$, $d^2+e^2 = 1$, then B is an orthogonal matrix.

If $a = a(t), b = b(t), c = c(t)$, then $B(t) = (a(t), b(t), c(t))$ is a curve on $SO(3)$.

Definition 2.3 *If $p \in \mathbb{R}^3$ is any point and $\delta(u)$ is a curve in $\mathbb{R}^3$, then* cone surface *is defined as*

$$K(u, v) = p + v\delta(u)$$

where $v \in \mathbb{R}$.

So we define cone surface of orthogonal curve by using the definitions of cone surface and eigencurve,

Definition 2.4 *If $C(t)$ is an eigencurve of $\alpha(t)$ orthogonal 1-parameter curve, then the surface*

$$K(v, t) = p + vC(t)$$

is called cone surface of orthogonal curve $\alpha(t)$.

Moreover, by using a regular curve in $\mathbb{R}^3$ we can find an orthogonal curve in $SO(3)$ [3]. So we have the following theorem.

Theorem 2.5 *If orthogonal matrix $A(t)$ is obtained with $\alpha(t)$ using Cayley's theorem and normed projection, then $\alpha(t) = C(t)$.*

So, there is a relation among unit sphere, S^2, $C^\infty(\mathbb{R}^3 - \{0\})$ and $\{\alpha(t) \mid \alpha(t) \subset SO(3)\}$ (Figure 1).

As an algorithmic;

1. *Given $\alpha(t) \subset \mathbb{R}^3$ a regular curve,*
2. *Let normed projection $\alpha(t)$ on S^2 be $\beta(t)$,*
3. *Unit vector $\beta(t)$ defines a curve on $\mathbb{R}^3$,*
4. *Characteristic vectors for $\lambda = 1$ causes a curve on $\mathbb{R}^3$.*

So we have the following commutative diagram .

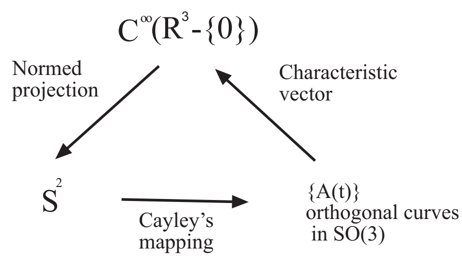

Figure 1: Commutative diagram

3 Matlab Applications

Let $\alpha(t)$ be a curve on $SO(3)$. Then

$$\alpha(t) = [\alpha_{ij}(t)], t \in \mathbb{R}$$

is an orthogonal matrix. For every t, $[\alpha_{ij}(t)]$ has an eigenvalue 1 and an eigenvector corresponding to this eigenvalue. Suppose that this eigenvector is $\overrightarrow{X(t)}$. We can define a curve in $\mathbb{R}^3$,

$$C(t) = X(t).$$

For example, for the curve

$$\alpha(t) = \frac{1}{1+t+t^2} \begin{bmatrix} -t & t+t^2 & 1+t \\ 1+t & -t & t+t^2 \\ t+t^2 & 1+t & -t \end{bmatrix}$$

,$t \in \mathbb{R}^+$, we have an eigencurve and cone surface as in Figure 2.

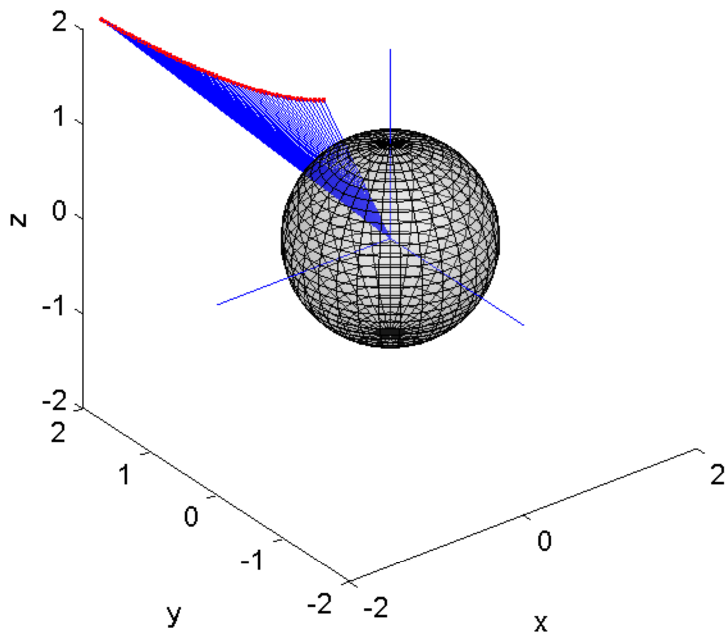

Figure 2: Cone surface and eigencurve of $\alpha(t)$

Example 3.1 *If we choose* $a = \frac{1}{\lambda}2\cos t$, $b = \frac{1}{\lambda}7\sin t$, $c = 4$, *where* $\lambda = \frac{1}{\sqrt{4\cos^2 t + 49\sin^2 t + 16}}$ *and* $d = \frac{1}{2}$, $e = \frac{\sqrt{3}}{2}$ *for* $B(t)$, *then we have Figure 3.*

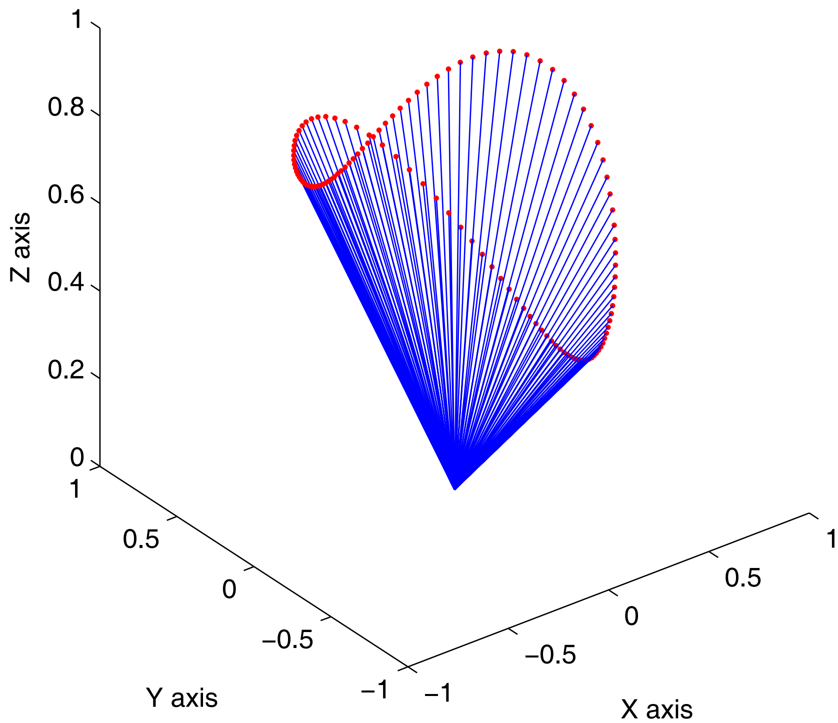

Figure 3: Cone surface and eigencurve of a spherical orthogonal curve

The Matlab m file is as follows.

```
clear all, close all, clc

 for t =1:3:360;
    cc=1;
    axis([-cc cc -cc cc 0 cc])
    xlabel('X axis')
    ylabel('Y axis')
    zlabel('Z axis')
```

```
    axis square
  A=2*cosd(t);
  B=7*sind(t);
  C=4;
    N =(A^2+B^2+C^2)^(1/2);
    a=A/N;
    b=B/N;
    c=C/N;
    Q=30
   u=  a*a*(1-cosd(Q))+cosd(Q)
   v=a*b*(1-cosd(Q))-c*sind(Q)
   w=a*c*(1-cosd(Q))+b*sind(Q);
    d=a*b*(1-cosd(Q))+c*sind(Q)
    e=b*b*(1-cosd(Q))+cosd(Q)
    f=b*c*(1-cosd(Q))-a*sind(Q);
    g=a*c*(1-cosd(Q))-b*sind(Q)
    h=b*c*(1-cosd(Q))+a*sind(Q)
    k=c*c*(1-cosd(Q))+cosd(Q);
A=[u v w;
    d e f;
    g h k]
z=2;
x=(((e-1)*(k-1)-h*f)/(d*h-g*(e-1)))*z
y=((d*(k-1)-g*f)/(g*(e-1)-d*h))*z
X=[x;y;z]
hold on
%plot3(x,y,z, 'r.')
[V,D]=eig(A)
plot3(V(1,3),V(2,3),V(3,3), 'r.')
line([0,V(1,3)],[0,V(2,3)],[0,V(3,3)])
pause(0.02)
 end
```

4 Conclusion

$\mathbb{R}^3$ is a differentiable manifold. $SO(3)$ is a differentiable manifold with dimensional 3. We can define a regular curve in $\mathbb{R}^3$ and $SO(3)$. The important relation can be established between the curves $\alpha(t) \subset SO(3)$ and $\beta(t) \subset \mathbb{R}^3$. If $\alpha(t) \subset SO(3)$ is a regular curve, then for all t , every $\alpha(t)$ is an orthogonal matrix and $\alpha(t)$ has an eigenvector for $\lambda = 1$ eigenvalue. By using this eigenvector we can define a line passing through the origin . All of these lines define a cone surface and are called cone surface of $\alpha(t)$ orthogonal curve. An algorithmic structure is observed in Matlab application.

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