International Mathematical Forum, Vol. 7, 2012, no. 39, 1905 - 1914

The Sequence Spaces $[\hat{w}(M, \Delta_u^v, q, s)]$ and $[\hat{w}(M, \Delta_u^v, q, s)]_\theta$ and Related Results

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Abstract

In this paper, we define sequence spaces : $[\hat{w}(M, \Delta_u^v, q, s)]$ and $[\hat{w}(M, \Delta_u^v, q, s)]_\theta$ and give some inclusion relations between these spaces and some related results.

Mathematics Subject Classification: 46A45, 40C05

Keywords: Sequence spaces, lacunary sequence, almost convergence and Orlicz function

1 Introduction

The spaces of lacunary strong convergence have been introduced by Freedman et al. [4]. A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

We recall that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

The Orlicz function M can always be represented in the following integral form (see Krasnoselskii and Rutickii [8])

$$M(x) = \int_0^x \varphi(t)dt,$$

where φ, known as the kernel of M, is right-differentiable for $t \geq 0, \varphi(0) = 0, \varphi(t) > 0$ for $t > 0, \varphi$ is nondecreasing and $\varphi(t) \to \infty$, as $t \to \infty$.

If convexity of M is replaced by $M(x+y) \leq M(x)+M(y)$, then it is called a modulus function, defined and discussed by Ruckle [14] and Maddox [11].

A sequence $x \in l_\infty$, the space of bounded sequences $x = (x_k)$, is said to be almost convergent to L (see [10]) if

$$\lim_{k\to\infty} t_{km}(x) = L, \text{ uniformly in } m, \text{ where}$$

$$t_{km}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}.$$

Using the concept of almost convergence, Das and sahoo [2] introduced the sequence spaces

$$\hat{w} = \{x = (x_k) : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} t_{km}(x-L) = 0, \text{uniformly in } m, \text{ for some } L\},$$

and

$$[\hat{w}] = \{x = (x_k) : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \mid t_{km}(x-L) \mid = 0, \text{uniformly in } m, \text{ for some } L\}.$$

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define what is called an Orlicz sequence space :

$$l_M := \{x = (x_k) : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) < \infty, \text{ for some } \rho > 0\}$$

which is a Banach space with the norm :

$$\parallel x \parallel_M = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) \leq 1\}.$$

Chishti [1] introduced the sequenc spaces :

For an Orlicz function M and some $\rho > 0$,

$$\begin{aligned} [\hat{w}(M)] &= \{x = (x_k) : \frac{1}{n+1}\sum_{k=0}^{n} M(\frac{\mid t_{km}(x-L) \mid}{\rho}) \to 0, \\ &\text{as } n \to \infty, \text{ uniformly in } m, \text{ for some } L\}, \end{aligned}$$

and

$$\begin{aligned}[\hat{w}(M)]_\theta &= \{x=(x_k): \sup_m \frac{1}{h_r}\sum_{k\in I_r} M(\frac{\mid t_{km}(x-L)\mid}{\rho}) \to 0,\\ \text{as } r &\to \infty,\ \text{for some } L\}.\end{aligned}$$

Now, if v is a nonnegative integer, $u=(u_k)$ is any sequence such that $u_k \neq 0$ for each $k, w(X)$ denotes the space of all sequences with elements in X, where (X,q) denotes a seminormed space, seminormed by q, and s is any real number such that $s \geq 0$, then we define the following sequence spaces :

$$\begin{aligned}[\hat{w}(M,\Delta_u^v,q,s)] &= \{x=(x_k): \frac{1}{n+1}\sum_{k=0}^{n} k^{-s}M(q(\frac{t_{km}(\Delta_u^v x-L)}{\rho})) \to 0,\\ \text{as } n &\to \infty,\ \text{uniformly in } m,\ \text{for some } L\},\end{aligned}$$

and

$$\begin{aligned}[\hat{w}(M,\Delta_u^v,q,s)]_\theta &= \{x=(x_k): \sup_m \frac{1}{h_r}\sum_{k\in I_r} k^{-s}M(q(\frac{t_{km}(\Delta_u^v x-L)}{\rho})) \to 0,\\ \text{as } r &\to \infty,\ \text{for some } L\},\end{aligned}$$

where

$$\Delta_u^0 x = u_k x_k,$$

$$\Delta_u^1 x = u_k x_k - u_{k+1}x_{k+1},$$

$$\Delta_u^2 x = \Delta(\Delta_u^1 x),$$

$$\vdots$$

$$\Delta_u^v x = \Delta(\Delta_u^{v-1} x),$$

so that

$$\Delta_u^\upsilon x = \Delta_{u_k}^\upsilon x_k = \sum_{r=0}^{\upsilon} (-1)^r \binom{\upsilon}{r} u_{k+r} x_{k+r}.$$

If $\upsilon = 0$, $\Delta x_k = x_k$ for all k, $u = e = (1, 1, 1, \cdots)$ and $s = 0$, then the above spaces reduce to those defined and studied by Chishti [1]. Also, we give the following definition :

Definition 1 *A sequence $x = (x_k)$ is said to be lacunary $[\hat{w}(M, \Delta_u^\upsilon, q, s)]$-convergent to L if*

$$\lim_{r\to\infty} \sup_m \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M(q(\frac{t_{km}(\Delta_u^\upsilon x - L)}{\rho})) = 0.$$

By $[\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta$, we denote the set of all lacunary $[\hat{w}(M, \Delta_u^\upsilon, q, s)]$-convergent sequences and we write $[\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta - \lim x = L$, for $x \in [\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta$

If $M(x) = x$, $\upsilon = 0$, $\Delta x_k = x_k$ for all k, $u = e$ and $s = 0$, then $[\hat{w}(M, \Delta_u^\upsilon, q, s)] = [\hat{w}]$ and $[\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta = [\hat{w}]_\theta$.

2 Main Results

In this section we prove the following theorems :

Theorem 2.1 *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r > 1$. Then $[\hat{w}(M, \Delta_u^\upsilon, q, s)] \subset [\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta$ and $[\hat{w}(M, \Delta_u^\upsilon, q, s)] - \lim x = [\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta - \lim x$.*

Proof. Let $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and therefore

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

This implies that

$$\begin{aligned} \frac{1}{k_r} \sum_{i=1}^{k_r} i^{-s} M(q(\frac{t_{im}(\Delta_u^\upsilon x - L)}{\rho})) &\geq \frac{1}{k_r} \sum_{i\in I_r} i^{-s} M(q(\frac{t_{im}(\Delta_u^\upsilon x - L)}{\rho})) \\ &\geq \frac{\delta}{1+\delta} \frac{1}{h_r} \sum_{i\in I_r} i^{-s} M(q(\frac{t_{im}(\Delta_u^\upsilon x - L)}{\rho})), \end{aligned}$$

and if $x \in [\hat{w}(M, \Delta_u^\upsilon, q, s)]$ with $[\hat{w}(M, \Delta_u^\upsilon, q, s)] - \lim x = L$, then it follows that $x \in [\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta$ with

$[\hat{w}(M, \Delta_u^\upsilon, q, s)]_\theta - \lim x = L$.

Theorem 2.2 *Let* $\theta=(k_r)$ *be a lacunary sequence with* $\limsup q_r<\infty$. *Then* $[\hat{w}(M,\Delta_u^v,q,s)]_\theta \subset [\hat{w}(M,\Delta_u^v,q,s)]$ *and* $[\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x = [\hat{w}(M,\Delta_u^v,q,s)] - \lim x$.

Proof. Let $x\in[\hat{w}(M,\Delta_u^v,q,s)]_\theta$ with $[\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x = L$. Then for $\epsilon>0$, there exists j_0 such that for every $j\geq j_0$ and all m,

$$g_{im}=\frac{1}{h_r}\sum_{i\in I_j} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x-L)}{\rho}))<\epsilon$$

,that is, we can find some positive constant C such that

$$g_{im}<C, \tag{1}$$

for all j and m. Now, $\limsup q_r<\infty$ implies that

$$\frac{h_r}{k_r}=1-\frac{k_{r-1}}{k_r}>1-\frac{1}{1+\delta}=\frac{\delta}{1+\delta}.$$

This implies that there exists some positive number K such that

$$q_r<K, \text{ for all } r\geq 1. \tag{2}$$

Therefore for $k_{r-1}<n\leq k_r$, we have by (2.1) and (2.2),

$$\begin{aligned}
\frac{1}{n+1}\sum_{i=0}^{n} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x-L)}{\rho})) &\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x-L)}{\rho}))\\
&= \frac{1}{k_{r-1}}\sum_{j=0}^{r}\sum_{i\in I_j} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x-L)}{\rho}))\\
&= \frac{1}{k_{r-1}}[\sum_{j=0}^{j_0}\sum_{j=j_0+1}^{r}]\sum_{i\in I_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x-L)}{\rho})\\
&\leq \frac{1}{k_{r-1}}(\sup_{l\leq p\leq j_0} g_{pm})k_{j_0}+\epsilon(k_r-k_{j_0})\frac{1}{k_{r-1}}\\
&\leq C\frac{k_{j_0}}{k_{r-1}}+\epsilon K.
\end{aligned}$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, we get that $x\in[\hat{w}(M,\Delta_u^v,q,s)]$ with $[\hat{w}(M,\Delta_u^v,q,s)]-\lim x=L$.

Theorem 2.3 *Let* $\liminf q_r \leq \limsup q_r < \infty$. *Then* $[\hat{w}(M,\Delta_u^v,q,s)] = [\hat{w}(M,\Delta_u^v,q,s)]_\theta$.

Proof. It follows from Theorem 2.1 and Theorem 2.2

Theorem 2.4 *Let* $x \in [\hat{w}(M,\Delta_u^v,q,s)] \cap [\hat{w}(M,\Delta_u^v,q,s)]_\theta$*. Then* $[\hat{w}(M,\Delta_u^v,q,s)] - \lim x = [\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x$ *and* $[\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x$ *is unique for any lacunary sequence* $\theta = (k_r)$.

Proof. Let $x \in [\hat{w}(M,\Delta_u^v,q,s)] \cap [\hat{w}(M,\Delta_u^v,q,s)]_\theta$ and $[\hat{w}(M,\Delta_u^v,q,s)] - \lim x = L, [\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x = L^{'}$.

Suppose that $L \neq L^{'}$. Then we see that

$$\begin{aligned} i^{-s}M(q(\frac{L-L^{'}}{\rho})) &\leq \frac{1}{h_r}\sum_{i\in I_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L)}{\rho})) \\ &+\frac{1}{h_r}\sum_{i\in I_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L^{'})}{\rho})), \text{ for each } m \\ &\leq \lim_{r\to\infty}\sup_m \frac{1}{h_r}\sum_{i\in I_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L)}{\rho})) + 0. \end{aligned}$$

Hence there exists r_0 such that for $r > r_0$, we have

$$\frac{1}{h_r}\sum_{i\in I_r} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L}{\rho})) > \frac{1}{2} i^{-s}M(q(\frac{L-L^{'}}{\rho})).$$

But $[\hat{w}(M,\Delta_u^v,q,s)] - \lim x = L$ implies that

$$0 \geq \lim\sup(\frac{h_r}{k_r}) i^{-s}M(q(\frac{L-L^{'}}{\rho})) \geq \lim\inf(\frac{h_r}{k_r}) i^{-s}M(q(\frac{L-L^{'}}{\rho})) \geq 0$$

and therefore $\lim q_r = 1$. Hence using Theorem 2.2, we conclude that $[\hat{w}(M,\Delta_u^v,q,s)]_\theta \subset [\hat{w}(M,\Delta_u^v,q,s)]$ and $[\hat{w}(M,\Delta_u^v,q,s)]_\theta - \lim x = L^{'} = L = [\hat{w}(M,\Delta_u^v,q,s)] - \lim x$.

Further,

$$\begin{aligned} &\frac{1}{n+1}\sum_{i=0}^{n} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L)}{\rho})) + \frac{1}{n+1}\sum_{i=0}^{n} i^{-s}M(q(\frac{t_{im}(\Delta_u^v x - L^{'})}{\rho})) \\ \geq\ & i^{-s}M(q(\frac{L-L^{'}}{\rho})) \geq 0 \end{aligned}$$

and taking the limit of both sides as $n \to \infty$, we see that $i^{-s}M(q(\frac{L-L^{'}}{\rho})) = 0$ and this shows that $L = L^{'}$ for any Orlicz function M.

Theorem 2.5 *Suppose that for a given* $\epsilon > 0$*, there exist* n_0 *and* m_0 *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s} M(q(\frac{t_{km}(\Delta_u^v x - L)}{\rho})) < \epsilon \text{ for all } n \geq n_0, m \geq m_0 \tag{3}$$

Then $x \in [\hat{w}(M, \Delta_u^v, q, s)]$.

Proof. Let $\epsilon > 0$ be given and choose $n_0^{'}$ and m_0 such that

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s} M(q(\frac{t_{km}(\Delta_u^v x - L)}{\rho})) < \frac{\epsilon}{4}$$

for $n \geq n_0$, $m \geq m_0$.

Now, it is enough to show that there exists $n_0^{''}$ such that for $n \geq n_0^{''}, 0 \leq m \leq m_0$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s} M(q(\frac{t_{km}(\Delta_u^v x - L)}{\rho})) < \epsilon.$$

Since m_0 is fixed, put $\sum_{k=0}^{m_0-1} \frac{1}{k} \sum_{j=0}^{m_0-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) = B$.

Now, let $0 \leq m \leq m_0$ and $n > m_0$, then

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} k^{-s} M(q(\frac{t_{km}(\Delta_u^v x - L)}{\rho})) &\leq \frac{1}{n}\sum_{k=0}^{m_0-1} \frac{1}{k}\sum_{j=0}^{m_0-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \\ &+ \frac{1}{n}\sum_{k=0}^{m_0-1} \mid \frac{1}{k}\sum_{j=m_0}^{m+k-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid \end{aligned}$$

$$+\frac{1}{n}\sum_{k=m_0}^{n-1} \frac{1}{k}\sum_{j=m}^{m+k-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \tag{4}$$

$$\leq \frac{B}{n} + \frac{1}{n}\sum_{k=0}^{m_0-1} \mid \frac{1}{k}\sum_{j=m_0}^{m_0+(k+m-m_0)-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid$$

$$+\frac{1}{n}\sum_{k=m_0}^{n-1} \mid \frac{1}{k}\sum_{j=m}^{m+k-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid .$$

Let $k - m_0 > n_0^{'}$. Then for $0 \leq m \leq m_0$, we have $k + m - m_0 \geq n_0^{'}$. Then from (2.3), we see that

$$\frac{1}{m_0}\sum_{k=0}^{m_0-1} \mid \frac{1}{k+m-m_0}\sum_{j=m_0}^{m_0+(k+m-m_0)-1} k^{-s} M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid < \frac{\epsilon}{4}. \tag{5}$$

From (2.4) and (2.5), we get that

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s}M(q(\frac{t_{km}(\Delta_u^v x - L)}{\rho})) \le \frac{B}{n} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon,$$

for sufficiently large n. Hence the result.

Theorem 2.6 *for every lacunary sequence* $\theta = (k_r)$, *we have* $[\hat{w}(M, \Delta_u^v, q, s)]_\theta \cap l_\infty = [\hat{w}(M, \Delta_u^v, q, s)]$.

Proof. Let $x \in [\hat{w}(M, \Delta_u^v, q, s)]_\theta \cap l_\infty$. Then for $\epsilon > 0$, there exist r_0 and p_0 such that

$$\frac{1}{h_r}\sum_{k=0}^{h_r-1} k^{-s}M(q(\frac{t_{kp}(\Delta_u^v x - L)}{\rho})) < \frac{\epsilon}{2} \tag{6}$$

for $r \ge r_0$ and $p \ge p_0$, $p = k_{r-1} + 1 + i, i \ge 0$.

Now, let $n \ge h_r$, m be an integergreater than or equal to 1. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s}M(q(\frac{t_{kp}(\Delta_u^v x - L)}{\rho})) \le \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{k=0}^{m-1} \mid \sum_{j=p+\mu h_r}^{p+(\mu+1)h_r-1} k^{-s}M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid$$

$$+\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{j=p+mh_r}^{m-1}\sum_{j=p+\mu h_r}^{p+k-1} k^{-s}M(q(\frac{\Delta_u^v x_j - L}{\rho})) \tag{7}$$

$$\begin{aligned} \le \quad & \frac{1}{n}\sum_{\mu=0}^{m-1}\sum_{k=\mu h_r}^{(\mu+1)h_r-1}\frac{1}{k} \mid \sum_{j=p}^{p+k-1} k^{-s}M(q(\frac{\Delta_u^v x_j - L}{\rho})) \mid \\ & +\frac{1}{n}\sum_{k=mh_r}^{n-1}\frac{1}{k}\sum_{j=p}^{p+k-1} k^{-s}M(q(\frac{\Delta_u^v x_j - L}{\rho})). \end{aligned}$$

Since $x \in l_\infty$, for all j, $M(q(\frac{\Delta_u^v x_j - L}{\rho})) < B$. So from (2.6) and (2.7), we have

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s}M(q(\frac{t_{kp}(\Delta_u^v x - L)}{\rho})) \le \frac{1}{n} m h_r \frac{\epsilon}{2} + \frac{Bh_r}{n}.$$

For $\frac{h_r}{n} \le 1$, $\frac{Bh_r}{n}$ can be made less than $\frac{\epsilon}{2}$ by taking n sufficiently large and since $\frac{mh_r}{n} \le 1$, then

$$\frac{1}{n}\sum_{k=0}^{n-1} k^{-s} M(q(\frac{t_{kp}(\Delta_u^v x - L)}{\rho})) < \epsilon$$

for $r \geq r_0, p \geq p_0$. Hence, by Theorem 2.5, we get that $[\hat{w}(M,\Delta_u^v,q,s)]_\theta \cap l_\infty \subset [\hat{w}(M,\Delta_u^v,q,s)]$. It is trivial that $[\hat{w}(M,\Delta_u^v,q,s)] \subset [\hat{w}(M,\Delta_u^v,q,s)]_\theta \cap l_\infty$. Hence the result.

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Received: March, 2012