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On Left Bicrossproduct Hopf Algebras[1]

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Abstract

Let $X = GM$ be a factorization of a group into a subgroup G and a subsemigroup M with identity and a left inverse property. A bialgebra $H = kM \bowtie k(G)$ with basis $m \otimes \delta_g$ where $m \in M$ and $g \in G$ is called a left Hopf algebra if there is a one-sided antipode map S such that $S(m \otimes \delta_g) = (m \triangleleft g)^L \otimes \delta_{(m \triangleright g)^{-1}}$. In this paper, we show that the quantum double $D(kM \bowtie k(G))$ can be generated by $H = kM \bowtie k(G)$ and its dual $H^* = k(M) \bowtie kG$ with specific cross relations. Moreover, an interesting example for these left Hopf algebras is introduced.

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1 Introduction

If we talk about non-commutative and non-cocommutative Hopf algebras we should mention the bicrossproducts that are associated to a factorization of groups. These bicrossproduct Hopf algebras can be applied in the quantum mechanics, geometry and the interrelation between them (see [9]). Many authors discussed and analyzed these algebras and their dual (see [1], [2] and [4]).

For a finite-dimensional Hopf algebra H, the quantum double is a Hopf algebra double-crossproduct $D(H) = H^{*op} \bowtie H$. More precisely, it is a Hopf algebra factorising into H^{*op} and H and given via a double-semidirect product by mutual coadjoint actions of these two factors on each other. This

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formulation is from [8] which is based on the work of Drinfeld [5].

Beggs et al. [3] computed the quantum double for the bicrossproduct Hopf algebras associated to a factorization of a finite group into two subgroups.

Green et al. [7] defined a left Hopf algebra to be a k-bialgebra $(B, m, \Delta, \eta, \epsilon : k)$ with a left antipode S, i.e., $S \in Hom_k(B, B)$ and $S * id = \eta\epsilon$.

In this paper, we continue what we have started in [6] and generalize some results of [3] using this definition of left Hopf algebras in a specific case. Explicitly, for a left Hopf algebra $H = kM \triangleright\!\blacktriangleleft k(G)$ associated to the factorization $X = GM$ of a group X into a subgroup G and a subsemigroup M with identity and left inverse property, we show that the quantum double $D(kM \triangleright\!\blacktriangleleft k(G))$ can be generated by $H = kM \triangleright\!\blacktriangleleft k(G)$ and its dual $H^* = k(M) \blacktriangleright\!\triangleleft kG$ with specific cross relations, where $k(G)$ is the Hopf algebra of function on G and kM is the semigroup left Hopf algebra of M.

2 Preliminaries and Definitions

In this section we present some basic concepts and definitions from [3], [6] and [8] that will be used later.

Let $X = GM$ be a group which factorizes into a subgroup G and a subsemigroup with identity M. Then M acts on G through the left action $\triangleright$: $M \times G \to G$ and G acts on M through the right action $\triangleleft$: $M \times G \to M$. These actions are defined by the unique factorization

$$mg = (m \triangleright g)(m \triangleleft g),$$

where $m \in M$ and $g \in G$. Comparing with [3], it can be easily shown that these actions satisfy the following equalities for all $m, m' \in M$ and $g, g' \in G$:

$$m \triangleleft e = m, (m \triangleleft g) \triangleleft g' = m \triangleleft (gg'); e \triangleleft g = e,$$

$$(mm') \triangleleft g = (m \triangleleft (m' \triangleright g))(m' \triangleleft g),$$

$$e \triangleright g = g, m \triangleright (m' \triangleright g) = (mm') \triangleright g; m \triangleright e = e,$$

$$m \triangleright (gg') = (m \triangleright g)((m \triangleleft g) \triangleright g').$$

We can associate to this factorization a bicrossproduct bialgebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ with basis $m \otimes \delta_g$ where $m \in M$ and $g \in G$. The product, unit, coproduct and counit are defined as follows:

$$(m \otimes \delta_g)(m^{'} \otimes \delta_{g'}) = \delta_{g,m' \triangleright g'}(mm^{'} \otimes \delta_{g'}),$$

$$1_H = \sum_g e \otimes \delta_g \,,$$

$$\Delta(m \otimes \delta_g) = \sum_{x,y \in G: xy=g} m \otimes \delta_x \otimes (m \triangleleft x) \otimes \delta_y \,,$$

$$\epsilon_H(m \otimes \delta_g) = \delta_{g,e} \,.$$

If there exists a left inverse $m^L \in M$ for each $m \in M$, then H becomes a left Hopf algebra and the left antipode will be given by:

$$S(m \otimes \delta_g) = (m \triangleleft g)^L \otimes \delta_{(m \triangleright g)^{-1}}.$$

It can be noted that $H = kM \triangleright\!\!\blacktriangleleft k(G)$ has the smash product algebra structure by the induced action of M and the smash coproduct coalgebra structure by the induced coaction of G.

For the notation $H = kM \triangleright\!\!\blacktriangleleft k(G)$, kM is the semigroup left Hopf algebra of the semigroup M with identity and the left inverse property. A basis of kM is given by the elements of M, with multiplication given by the semigroup product in M, and comultiplication given by $\Delta m = m \otimes m$ for $m \in M$. Also, $k(G)$ is the Hopf algebra of functions on G with basis given by δ_g for $g \in G$. The product is just multiplication of functions, and the coproduct is

$$\Delta \delta_g = \sum_{x,y \in G: xy=g} \delta_x \otimes \delta_y.$$

Moreover, a dual bicrossproduct bialgebra $H^* = k(M) \blacktriangleright\!\!\triangleleft kG$ can be defined with basis $\delta_m \otimes g$ where $m \in M$ and $g \in G$. The product, unit, coproduct and counit are defined as follows:

$$(\delta_m \otimes g)(\delta_{m'} \otimes g^{'}) = \delta_{m \triangleleft g, m'}(\delta_m \otimes gg^{'}),$$

$$1_{H^*} = \sum_m \delta_m \otimes e\,,$$

$$\Delta(\delta_m \otimes g) = \sum_{a,b \in M: ab=m} \big(\delta_a \otimes (b \triangleright g)\big) \otimes (\delta_b \otimes g)\,,$$

$$\epsilon_{H^*}(\delta_m \otimes g) = \delta_{m,e}\,.$$

If there exists a left inverse $m^L \in M$ for each $m \in M$, then H^* becomes a left Hopf algebra and the left antipode will be given by:

$$S(\delta_m \otimes g) = \delta_{(m \triangleleft g)^L} \otimes (m \triangleright g)^{-1}\,.$$

Definition 2.1 *Let $X = GM$ be a factorization of a group into a subgroup G and a subsemigroup M with identity and a left inverse property. A bialgebra $H = kM \bowtie k(G)$ with basis $m \otimes \delta_g$ where $m \in M$ and $g \in G$ is called a left Hopf algebra if there is a left antipode S such that*

$$S(m \otimes \delta_g) = (m \triangleleft g)^L \otimes \delta_{(m \triangleright g)^{-1}}.$$

Definition 2.2 *Given any two groups X and Y (not necessarily subgroups of a given group) and a group homomorphism $\varphi : Y \to Aut(X)$, the new group $X \rtimes_\varphi Y$ (or simply $X \times Y$) is called the semidirect product of X and Y with respect to φ with an operation * defined by*

$$(x_1, y_1) * (x_2, y_2) = (x_1\varphi(y_1)(x_2), y_1y_2)$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

3 Example

Here we try to illustrate the concept of the generalized bicrossproduct bialgebras and the left bicrossproduct Hopf algebras by the following example:

Example 3.1 *Consider the set $X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R}, ad-bc = 1 \right\}$. It easy to check that X is an associative group under the usual matrices multiplication with identity equal to the usual 2×2 identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.*

For any element $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in X$, *the inverse is* $A^{-1} = \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \in X$.

Now, we take a subset G *of* X *to be* $G = \left\{ \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \mid u, v \in \mathbb{R}, u^2+v^2 = 1 \right\}$. *One can check that* G *is a subgroup of* X. *Indeed if* $U = \begin{pmatrix} u_1 & -v_1 \\ v_1 & u_1 \end{pmatrix}$ *and* $V = \begin{pmatrix} u_2 & -v_2 \\ v_2 & u_2 \end{pmatrix}$ *are elements in* G, *then*

$$\begin{aligned} U \cdot V = & \begin{pmatrix} u_1 & -v_1 \\ v_1 & u_1 \end{pmatrix} \cdot \begin{pmatrix} u_2 & -v_2 \\ v_2 & u_2 \end{pmatrix} \\ = & \begin{pmatrix} u_1u_2 - v_1v_2 & -u_1v_2 - v_1u_2 \\ u_2v_1 + u_1v_2 & -v_1v_2 + u_1u_2 \end{pmatrix} \\ = & \begin{pmatrix} u_1u_2 - v_1v_2 & -(v_1u_2 + u_1v_2) \\ v_1u_2 + u_1v_2 & u_1u_2 - v_1v_2 \end{pmatrix}, \end{aligned}$$

with $u_1^2 + v_1^2 = 1$ *and* $u_2^2 + v_2^2 = 1$. *To verify that* $U \cdot V$ *is in* G, *we calculate*

$$\begin{aligned} (u_1u_2 - v_1v_2)^2 + (v_1u_2 + u_1v_2)^2 &= u_1^2u_2^2 + v_1^2v_2^2 + v_1^2u_2^2 + u_1^2v_2^2 \\ &= u_1^2(u_2^2 + v_2^2) + v_1^2(u_2^2 + v_2^2) \\ &= (u_1^2 + v_1^2)(u_2^2 + v_2^2) = 1 \cdot 1 = 1. \end{aligned}$$

So, G *is closed under the usual matrices multiplication. Next, for an element* $\acute{U} = \begin{pmatrix} \acute{u} & -\acute{v} \\ \acute{v} & \acute{u} \end{pmatrix} \in G$ *the inverse element is* $\acute{U}^{-1} = \frac{1}{\acute{u}^2+\acute{v}^2} \begin{pmatrix} \acute{u} & \acute{v} \\ -\acute{v} & \acute{u} \end{pmatrix} = \begin{pmatrix} \acute{u} & \acute{v} \\ -\acute{v} & \acute{u} \end{pmatrix} \in G$.

Now, we take the subset M *of* X *to be* $M = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R}, t \geq 0 \right\}$. *We show that* M *is a semigroup of* X *with identity. Let* $S = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix}$ *be elements of* M. *Then*

$$S \cdot T = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_1 + t_2 & 1 \end{pmatrix},$$

with $t_1 \geq 0$ *and* $t_2 \geq 0$ *that implies* $t_1+t_2 \geq 0$. *Hence,* $S \cdot T \in M$. *The identity matrix* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is in* M. *However,* M *does not have an inverse element for each* $\acute{S} \in M$. *So* M *is not a subgroup of* X.

Next, if we take $U = \begin{pmatrix} u_1 & -v_1 \\ v_1 & u_1 \end{pmatrix} \in G$ *and* $S = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \in M$*, then*

$$\begin{aligned} U \cdot S &= \begin{pmatrix} u_1 & -v_1 \\ v_1 & u_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 - v_1 t_1 & -v_1 \\ v_1 + u_1 t_1 & u_1 \end{pmatrix} \in X, \end{aligned}$$

where ${u_1}^2 + {v_1}^2 = 1$, $t_1 > 0$ *and* $u_1, v_1, t_1 \in \mathbb{R}$. *It is clear that* $G \cdot M \subseteq X$. *Now we can forget our original group* X *and put* $\acute{X} = G \cdot M$. *It can be noted that* $\acute{X}$ *is just a subset of* X *and not necessarily a subgroup. So we can construct a generalized bicrossproduct bialgebra form* G *and* M *to be* $H = \mathbb{R}M \blacktriangleright\!\!\triangleleft \mathbb{R}(G)$. *If, somehow, we can define a left antipode* $\mathbf{S}$ *on the generalized bicrossproduct bialgebra* $H = \mathbb{R}M \blacktriangleright\!\!\triangleleft \mathbb{R}(G)$*, then the left bicrossproduct Hopf algebra is obtained.*

4 The Quantum Double

For a finite-dimensional left Hopf algebra H, the quantum double is a left Hopf algebra double-crossproduct $D(H) = H^{*op} \bowtie H$. More precisely, it is a left Hopf algebra factorising into H^{*op} and H and given via a double-semidirect product by mutual coadjoint actions of these two factors on each other. This formulation is based on the work of Majid [8] which is itself based on the work of Drinfeld [5]. More specifically, it is built on $H^* \otimes H$ as a linear space with product

$$(\widehat{h}^* \otimes h)(\widetilde{h}^* \otimes h') = \Sigma \langle S\widetilde{h}^*_{(1)}, h_{(1)} \rangle \widetilde{h}^*_{(2)} \widehat{h}^* \otimes h_{(2)} h' \langle \widetilde{h}^*_{(3)}, h_{(3)} \rangle \qquad (1)$$

$$\widehat{h}^*, \widetilde{h}^* \in H^*, h, h' \in H.$$

The formulas for tensor product unit, counit, coproduct, and antipode are given by:

$$\begin{aligned} 1_{D(H)} &= 1_{H^*} \otimes 1_H, \\ \epsilon_{D(H)} &= \epsilon_{H^*} \otimes \epsilon_H, \\ \Delta_{D(H)} &= (id \otimes \tau \otimes id) \circ (\Delta \circ \Delta), \\ S_{D(H)}(\widehat{h}^* \otimes h) &= (1 \otimes Sh)(S^{-1}\widehat{h}^* \otimes 1), \end{aligned}$$

where $\tau(\widehat{h}^* \otimes h) = h \otimes \widehat{h}^*$. The Hopf algebra structure takes the double cross product from the formula

$$(\widehat{h}^* \otimes h)(\widetilde{h}^* \otimes h') = \sum (h_{(1)} \triangleright \widetilde{h}^*_{(1)}) \widehat{h}^* \otimes (h_{(2)} \triangleleft \widetilde{h}^*_{(2)}) h'. \qquad (2)$$

The required mutual coadjiont actions are defined by

$$h \triangleright \widehat{h}^* = \sum \widehat{h}^*_{(2)} \langle h, \big(S\widehat{h}^*_{(1)}\big)\widehat{h}^*_{(3)}\rangle, \tag{3}$$

$$h \triangleleft \widehat{h}^* = \sum h_{(2)} \langle \widehat{h}^*, \big(Sh_{(1)}\big)h_{(3)}\rangle, \tag{4}$$

where $\widehat{h}^* \in H^*$ and $h \in H$.

It can be noted that in all these formulas, the expressions are given in terms of the Hopf algebras H and H^*. Also, for $h \in H$ and $\widehat{h}^* \in H^*$, we have $\langle S(h), S(\widehat{h}^*)\rangle = \langle h, \widehat{h}^*\rangle$.

Proposition 4.1 *Let $H = kM \bowtie k(G)$ be a left Hopf algebra associated to a factorization of a group $X = GM$ into a subgroup G and a subsemigroup M with identity and a left inverse property. Then the formula for the action of $H = kM \bowtie k(G)$ on its dual $H^* = k(M) \bowtie kG$ is given by*

$$(m_1 \otimes \delta_{g_1}) \triangleright (\delta_m \otimes g) = \delta_{g,(m\triangleright g)g_1}\big(\delta_{m_1' m {m_1'}^L} \otimes m_1' \triangleright g\big),$$

where $m, m_1 \in M$, $g, g_1 \in G$, $m_1 \otimes \delta_{g_1} \in H$, $\delta_m \otimes g \in H^*$ and $m_1' = m_1 \triangleleft (m \triangleright g)^{-1}$.

Proof. Let $\widehat{h}^* = \delta_m \otimes g \in H^*$ and $h = m_1 \otimes \delta_{g_1}$, we calculate

$$\begin{aligned}(\Delta \otimes id)\Delta(\delta_m \otimes g) &= (\Delta \otimes id) \sum_{a,b\in M: ab=m} \Big(\delta_a \otimes (b \triangleright g)\Big) \otimes \Big(\delta_b \otimes g\Big) \\ &= \sum_{a,b\in M: ab=m} \Delta\Big(\delta_a \otimes (b \triangleright g)\Big) \otimes id(\delta_b \otimes g) \\ &= \sum_{a,b\in M: ab=m} \Big(\sum_{c,d\in M: cd=a} \big(\delta_c \otimes (d \triangleright (b \triangleright g))\big) \otimes \big(\delta_d \otimes (b \triangleright g)\big)\Big) \otimes (\delta_b \otimes g) \\ &= \sum_{c,d,b\in M: cdb=m} \big(\delta_c \otimes ((db) \triangleright g)\big) \otimes \big(\delta_d \otimes (b \triangleright g)\big) \otimes (\delta_b \otimes g).\end{aligned}$$

This yields

$$\widehat{h}^*_{(1)} = \delta_c \otimes (db \triangleright g),$$

$$\widehat{h}^*_{(2)} = \delta_d \otimes (b \triangleright g),$$

$$\widehat{h}^*_{(3)} = \delta_b \otimes g.$$

Now, if we substitute these values into equation (3) we get $h{\triangleright}\widehat{h}^* = \sum \widehat{h}^*_{(2)}\langle h, (S\widehat{h}^*_{(1)})\widehat{h}^*_{(3)}\rangle$, or equivalently

$$\begin{aligned}(m_1 \otimes \delta_{g_1}) \triangleright (\delta_m \otimes g) &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\langle m_1 \otimes \delta_{g_1}, S(\delta_c \otimes (db \triangleright g))(\delta_b \otimes g)\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\Big\langle m_1 \otimes \delta_{g_1}, \Big(\delta_{(c\triangleleft(db\triangleright g))^L} \otimes (c \triangleright (db \triangleright g))^{-1}\Big)(\delta_b \otimes g)\Big\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\Big\langle m_1 \otimes \delta_{g_1}, \Big(\delta_{c^L\triangleleft(m\triangleright g)} \otimes (m \triangleright g)^{-1}\Big)(\delta_b \otimes g)\Big\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\Big\langle m_1 \otimes \delta_{g_1}, \delta_{(c^L\triangleleft(m\triangleright g))\triangleleft(m\triangleright g)^{-1},b}\Big(\delta_{c^L\triangleleft(m\triangleright g)} \otimes (m \triangleright g)^{-1}g\Big)\Big\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\Big\langle m_1 \otimes \delta_{g_1}, \delta_{c^L,b}\Big(\delta_{c^L\triangleleft(m\triangleright g)} \otimes (m \triangleright g)^{-1}g\Big)\Big\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\, \delta_{c^L,b}\Big\langle m_1 \otimes \delta_{g_1}, \Big(\delta_{c^L\triangleleft(m\triangleright g)} \otimes (m \triangleright g)^{-1}g\Big)\Big\rangle \\ &= \sum_{cdb=m} (\delta_d \otimes (b \triangleright g))\, \delta_{c^L,b}\, \delta_{m_1,c^L\triangleleft(m\triangleright g)}\, \delta_{g_1,(m\triangleright g)^{-1}g}.\end{aligned}$$

To have a non-trivial answer we must have $c^L = b$. So we get

$$(m_1 \otimes \delta_{g_1}) \triangleright (\delta_m \otimes g) = \delta_{g_1,(m\triangleright g)^{-1}g} \sum_{cdc^L=m} (\delta_d \otimes (c^L \triangleright g))\delta_{m_1,c^L\triangleleft(m\triangleright g)}. \tag{5}$$

To obtain the desired action we solve for c which is fixed by the delta function inside the summation. We should have $m_1 = c^L \triangleleft (m \triangleright g)$, or equivalently

$$m_1 \triangleleft (m \triangleright g)^{-1} = c^L \triangleleft (m \triangleright g) \triangleleft (m \triangleright g)^{-1},$$

which implies

$$m_1 \triangleleft (m \triangleright g)^{-1} = c^L = m_1'.$$

Also, we have $cdc^L = m$ which can recalculated to be $d = c^L mc = m_1' m m_1'^L$. Therefore, equation (5) can be rewritten as

$$\begin{aligned}(m_1 \otimes \delta_{g_1}) \triangleright (\delta_m \otimes g) &= \delta_{g_1,(m\triangleright g)^{-1}g}(\delta_d \otimes (c^L \triangleright g) \\ &= \delta_{g_1,(m\triangleright g)^{-1}g}(\delta_{m_1' m m_1'^L} \otimes (m_1' \triangleright g)) \\ &= \delta_{g,(m\triangleright g)g_1}\big(\delta_{m_1' m m_1'^L} \otimes m_1' \triangleright g\big),\end{aligned}$$

as required. □

Proposition 4.2 *Let $H = kM \bowtie k(G)$ be a left Hopf algebra associated to a factorization of a group $X = GM$ into a subgroup G and a subsemigroup M*

with identity and a left inverse property. Then the formula for the action of $H^* = k(M) \blacktriangleright\!\!\triangleleft\, kG$ *on its dual* $H = kM \triangleright\!\!\blacktriangleleft k(G)$ *is given by*

$$(m_1 \otimes \delta_{g_1}) \triangleleft (\delta_m \otimes g) = \delta_{m \triangleleft g, {m'_1}^L (m_1 \triangleleft g_1)} \big(m'_1 \otimes \delta_{(m \triangleright g) g_1 g^{-1}} \big),$$

where $m, m_1 \in M$*,* $g, g_1 \in G$*,* $m_1 \otimes \delta_{g_1} \in H$*,* $\delta_m \otimes g \in H^*$ *and* $m'_1 = m_1 \triangleleft (m \triangleright g)^{-1}$.

Proof. Let $\widehat{h}^* = \delta_m \otimes g \in H^*$ and $h = m_1 \otimes \delta_{g_1} \in H$, we calculate $(\Delta \otimes id)\Delta(h)$ as follows:

$$\begin{aligned}
(\Delta \otimes id)\Delta(h) &= (\Delta \otimes id)\Delta(m_1 \otimes \delta_{g_1}) \\
&= (\Delta \otimes id)\Big(\sum_{x,y \in G : xy = g_1} (m_1 \otimes \delta_x) \otimes ((m_1 \triangleleft x) \otimes \delta_y) \Big) \\
&= \sum_{x,y \in G : xy = g_1} \Delta(m_1 \otimes \delta_x) \otimes id\big((m_1 \triangleleft x) \otimes \delta_y\big) \\
&= \sum_{x,y \in G : xy = g_1} \Big(\sum_{w,z \in G : wz = x} (m_1 \otimes \delta_w) \otimes (m_1 \triangleleft w) \otimes \delta_z \Big) \otimes (m_1 \triangleleft x) \otimes \delta_y \\
&= \sum_{wzy = g_1} (m_1 \otimes \delta_w) \otimes \big((m_1 \triangleleft w) \otimes \delta_z\big) \otimes \big((m_1 \triangleleft wz) \otimes \delta_y\big).
\end{aligned}$$

From the last equation we get the following summands:

$$h_{(1)} = m_1 \otimes \delta_w,$$

$$h_{(2)} = (m_1 \triangleleft w) \otimes \delta_z,$$

$$h_{(3)} = (m_1 \triangleleft wz) \otimes \delta_y.$$

Now, if we substitute these values into equation (4) we get $h \triangleleft \widehat{h}^* = \sum h_{(2)} \langle (S h_{(1)}) h_{(3)}, \widehat{h}^* \rangle$, or equivalently

$$\begin{aligned}(m_1 \otimes \delta_{g_1}) \lhd (\delta_m \otimes g) &= \sum_{wzy=g_1} ((m_1 \lhd w) \otimes \delta_z)\langle S(m_1 \otimes \delta_w)(m_1 \lhd wz \otimes \delta_y), \delta_m \otimes g\rangle \\
&= \sum_{wzy=g_1} ((m_1 \lhd w) \otimes \delta_z)\Big\langle \Big((m_1 \lhd w)^L \otimes \delta_{(m_1 \rhd w)^{-1}}\Big) \\
&\qquad \Big((m_1 \lhd wz) \otimes \delta_y\Big), \delta_m \otimes g\Big\rangle \\
&= \sum_{wzy=g_1} ((m_1 \lhd w) \otimes \delta_z)\Big\langle \delta_{(m_1 \rhd w)^{-1},(m_1 \lhd wz) \rhd y} \\
&\qquad \Big((m_1 \lhd w)^L (m_1 \lhd wz) \otimes \delta_y\Big), \delta_m \otimes g\Big\rangle \\
&= \sum_{wzy=g_1} ((m_1 \lhd w) \otimes \delta_z)\delta_{(m_1 \rhd w)^{-1},(m_1 \lhd wz) \rhd y} \\
&\qquad \Big\langle (m_1 \lhd w)^L (m_1 \lhd wz) \otimes \delta_y, \delta_m \otimes g\Big\rangle \\
&= \sum_{wzy=g_1} ((m_1 \lhd w) \otimes \delta_z)\delta_{(m_1 \rhd w)^{-1},(m_1 \lhd wz) \rhd y}\delta_{(m_1 \lhd w)^L (m_1 \lhd wz),m}\delta_{y,g}.\end{aligned} \tag{6}$$

To have a non-trivial answer we must have $y = g$. Consequently, we have $wzg = g_1$ which implies $wz = g_1 g^{-1}$. Hence, the last form of equation (6) can be rewritten as

$$(m_1 \otimes \delta_{g_1}) \lhd (\delta_m \otimes g) = \sum_{wz=g_1 g^{-1}} ((m_1 \lhd w) \otimes \delta_z)\delta_{(m_1 \rhd w)^{-1},(m_1 \lhd g_1 g^{-1}) \rhd g}\delta_{(m_1 \lhd w)^L (m_1 \lhd g_1 g^{-1}),m}. \tag{7}$$

Next we solve these equations for w and z. We need the following calculations for the double-crossproduct groups:

$$\begin{aligned}(m_1 \lhd w) \rhd w^{-1} &= (m_1 \rhd (ww^{-1}))(m_1 \rhd w)^{-1} \\
&= (m_1 \rhd e)(m_1 \rhd w)^{-1} \\
&= e(m_1 \rhd w)^{-1} \\
&= (m_1 \rhd w)^{-1}.\end{aligned} \tag{8}$$

Next,

$$\begin{aligned}m_1^L \lhd (m_1 \rhd w) &= ((m_1^L m_1) \lhd w)(m_1 \lhd w)^L \\
&= (e \lhd w)(m_1 \lhd w)^L \\
&= e(m_1 \lhd w)^L = (m_1 \lhd w)^L.\end{aligned} \tag{9}$$

Now, by the Kronecker map we have: $(m_1 \triangleright w)^{-1} = (m_1 \triangleleft g_1g^{-1}) \triangleright g$ and $m = (m_1 \triangleleft w)^L(m_1 \triangleleft g_1g^{-1})$. Using equation (8), we get $(m_1 \triangleleft w) \triangleright w^{-1} = (m_1 \triangleleft g_1g^{-1}) \triangleright g$, or equivalently

$$\begin{aligned} w^{-1} &= (m_1 \triangleleft w)^L \triangleright ((m_1 \triangleleft g_1g^{-1}) \triangleright g) \\ &= (m_1 \triangleleft w)^L(m_1 \triangleleft g_1g^{-1}) \triangleright g \\ &= m \triangleright g. \end{aligned}$$

Thus

$$w = (m \triangleright g)^{-1},$$

$$wz = g_1g^{-1} \Rightarrow z = w^{-1}g_1g^{-1} = (m \triangleright g)g_1g^{-1},$$

and

$$m_1 \triangleleft w = m_1 \triangleleft (m \triangleright g)^{-1} = m_1'.$$

The second delta function in the summation of equation (7) can be simplified noting that

$$(m_1 \triangleleft w)^L(m_1 \triangleleft g_1g^{-1}) = m_1'^L(m_1 \triangleleft g_1) \triangleleft g^{-1}.$$

Therefore,

$$\begin{aligned} (m_1 \otimes \delta_{g_1}) \triangleleft (\delta_m \otimes g) &= \sum_{wz=g_1g^{-1}} \big((m_1 \triangleleft w) \otimes \delta_z\big)\delta_{(m_1 \triangleright w)^{-1},(m_1 \triangleleft g_1g^{-1}) \triangleright g}\delta_{(m_1 \triangleleft w)^L(m_1 \triangleleft g_1g^{-1}),m} \\ &= \sum_{wz=g_1g^{-1}} \big(m_1' \otimes \delta_{(m \triangleright g)g_1g^{-1}}\big)\delta_{(m_1 \triangleright w)^{-1},(m_1 \triangleleft g_1g^{-1}) \triangleright g}\delta_{m_1'^L(m_1 \triangleleft g_1) \triangleleft g^{-1},m} \end{aligned}$$

or equivalently

$$\begin{aligned} (m_1 \otimes \delta_{g_1}) \triangleleft (\delta_m \otimes g) &= m_1' \otimes \delta_{(m \triangleright g)g_1g^{-1}}\delta_{m \triangleleft g,m_1'^L(m_1 \triangleleft g_1)} \\ &= \delta_{m \triangleleft g,m_1'^L(m_1 \triangleleft g_1)}\big(m_1' \otimes \delta_{(m \triangleright g)g_1g^{-1}}\big), \end{aligned}$$

as required. □

Theorem 4.3 *Let $H = kM \bowtie k(G)$ be a left Hopf algebra associated to a factorization of a group $X = GM$ into a subgroup G and a subsemigroup M with identity and a left inverse property. Then the quantum double $D(kM \triangleright\!\!\blacktriangleleft k(G))$ is generated by $H = kM \triangleright\!\!\blacktriangleleft k(G)$ and $H^* = k(M) \blacktriangleright\!\!\triangleleft kG$ with cross relations defined by the product*

$$(1 \otimes m_1 \otimes \delta_{g_1})(\delta_m \otimes g \otimes 1) = \delta_{m_1' m (m_1 \triangleleft g_1g^{-1})^L} \otimes (m_1 \triangleleft g_1g^{-1}) \triangleright g \otimes m_1' \otimes \delta_{(m \triangleright g)g_1g^{-1}},$$

where $m, m_1 \in M$, $g, g_1 \in G$ and $m_1' = m_1 \triangleleft (m \triangleright g)^{-1}$.

Proof. Let $h = m_1 \otimes \delta_{g_1} \in H$ and $\widehat{h}^* = \delta_m \otimes g \in H^*$. We want to find out $(1 \otimes h)(\widehat{h}^* \otimes 1)$ using equation (2). To do so, we need the following calculations:

$$\Delta(m_1 \otimes \delta_{g_1}) = \sum_{x,y \in G: xy = g_1} m_1 \otimes \delta_x \otimes (m_1 \lhd x) \otimes \delta_y,$$

and

$$\Delta(\delta_m \otimes g) = \sum_{w,z \in M: wz = m} \delta_w \otimes (z \rhd g) \otimes \delta_z \otimes g.$$

These yield $h_{(1)} = m_1 \otimes \delta_x$, $h_{(2)} = (m_1 \lhd x) \otimes \delta_y$, $\widehat{h}^*_{(1)} = \delta_w \otimes (z \rhd g)$ and $\widehat{h}^*_{(2)} = \delta_z \otimes g$. Applying proposition 4.1, we obtain

$$\begin{aligned} h_{(1)} \rhd \widehat{h}^*_{(1)} &= (m_1 \otimes \delta_x) \rhd (\delta_w \otimes (z \rhd g)) \\ &= \delta_{z \rhd g, (w \rhd (z \rhd g))x} \big(\delta_{m'_1 w m'^L_1} \otimes m'_1 \rhd (z \rhd g)\big) \\ &= \delta_{z \rhd g, (wz \rhd g)x} (\delta_{m'_1 w m'^L_1} \otimes (m'_1 z \rhd g)) \\ &= \delta_{z \rhd g, (m \rhd g)x} (\delta_{m'_1 w m'^L_1} \otimes (m'_1 z \rhd g)), \end{aligned}$$

where $m'_1 = m_1 \lhd (w \rhd (z \rhd g))^{-1} = m_1 \lhd ((wz) \rhd g))^{-1} = m_1 \lhd (m \rhd g)^{-1}$, since $wz = m$. Next, applying proposition 4.2 gives

$$\begin{aligned} h_{(2)} \lhd \widehat{h}^*_{(2)} &= \big((m_1 \lhd x) \otimes \delta_y\big) \lhd (\delta_z \otimes g) \\ &= (m_1 \lhd x) \lhd (z \rhd g)^{-1} \otimes \delta_{(z \rhd g)yg^{-1}} \delta_{(m_1 \lhd x) \lhd y, (m_1 \lhd x)(z \lhd g)} \\ &= (m_1 \lhd x) \lhd (z \rhd g)^{-1} \otimes \delta_{(z \rhd g)yg^{-1}} \delta_{(m_1 \lhd xy), (m_1 \lhd x)(z \lhd g)} \\ &= (m_1 \lhd x) \lhd (z \rhd g)^{-1} \otimes \delta_{(z \rhd g)yg^{-1}} \delta_{(m_1 \lhd g_1), (m_1 \lhd x)(z \lhd g)} \\ &= \delta_{m_1 \lhd g_1, (m_1 \lhd x)(z \lhd g)} \big((m_1 \lhd x) \lhd (z \rhd g)^{-1}\big) \otimes \delta_{(z \rhd g)yg^{-1}} \\ &= \delta_{m_1 \lhd g_1, (m_1 \lhd x)(z \lhd g)}\, m''_1 \otimes \delta_{(z \rhd g)yg^{-1}}, \end{aligned}$$

where $m''_1 = (m_1 \lhd x) \lhd (z \rhd g)^{-1}$. Now, formula (2) and these calculations give

$$\begin{aligned} (1 \otimes m_1 \otimes \delta_{g_1})(\delta_m \otimes g \otimes 1) &= \sum (h_{(1)} \rhd \widetilde{h}^*_{(1)}) 1 \otimes (h_{(2)} \lhd \widetilde{h}^*_{(2)}) 1 \\ &= \sum_{xy = g_1, wz = m} \delta_{z \rhd g, (m \rhd g)x} \delta_{m_1 \lhd g_1, (m_1 \lhd x)(z \lhd g)} \\ &\qquad \big(\delta_{m'_1 w m'^L_1} \otimes (m'_1 z \rhd g) \otimes m''_1 \otimes \delta_{(z \rhd g)yg^{-1}}\big). \end{aligned} \tag{10}$$

The delta functions in equation (10) imply that $z \rhd g = (m \rhd g)x$ and $m_1 \lhd g_1 =$

$(m_1 \triangleleft x)(z \triangleleft g)$. To simplify equation (10), we calculate the following:

$$\begin{aligned} m_1' z \triangleleft g &= (m_1' \triangleleft (z \triangleright g))(z \triangleleft g) \\ &= \Big(\big(m_1 \triangleleft (m \triangleright g)^{-1}\big) \triangleleft (z \triangleright g)\Big)(z \triangleleft g) \\ &= \big(m_1 \triangleleft (m \triangleright g)^{-1}(z \triangleright g)\big)(z \triangleleft g) \\ &= \big(m_1 \triangleleft (m \triangleright g)^{-1}(m \triangleright g)x\big)(z \triangleleft g) \\ &= (m_1 \triangleleft x)(z \triangleleft g) = m_1 \triangleleft g_1, \end{aligned}$$

or equivalently,

$$m_1' z = (m_1 \triangleleft g_1) \triangleleft g^{-1} = m_1 \triangleleft g_1 g^{-1}. \tag{11}$$

Also, as $z \triangleright g = (m \triangleright g)x$ and $xy = g_1$, we have

$$(z \triangleright g)y = (m \triangleright g)xy = (m \triangleright g)g_1, \tag{12}$$

and

$$m_1'' = (m_1 \triangleleft x) \triangleleft (z \triangleright g)^{-1} = m_1 \triangleleft x(z \triangleright g)^{-1} = m_1 \triangleleft (m \triangleright g)^{-1} = m_1'. \tag{13}$$

Finally, as $wz = m$, we have $m_1' w m_1'^{L} = m_1' m (m_1 \triangleleft g_1 g^{-1})^L$. Substituting these values in the right hand side of equation (10) gives

$$(1 \otimes m_1 \otimes \delta_{g_1})(\delta_m \otimes g \otimes 1) = \delta_{m_1' m (m_1 \triangleleft g g_1^{-1})^L} \otimes (m_1 \triangleleft g_1 g^{-1}) \triangleright g \otimes m_1' \otimes \delta_{(m \triangleright g) g_1 g^{-1}},$$

as required. □

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