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Tripled Coincidence Points for Monotone Operators in Partially Ordered Metric Spaces

Saud M. Alsulami and Abdullah Alotaibi

Department of Mathematics, King Abdulaziz University
P.O. Box: 80203 Jeddah 21589, Saudi Arabia
alsulami@kau.edu.sa, aalotaibi@kau.edu.sa

Abstract. Using the notion of compatible mappings in the setting of a partially ordered metric space, we prove the existence and uniqueness of tripled coincidence points involving a (ϕ, ψ)-contractive condition for a mappings having the mixed g-monotone property. We illustrate our results with the help of an example.

Keywords: Tripled coincidence point, partially ordered metric space, mixed g-monotone property

1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem. Afterward many authors obtained many important extensions of this principle (cf. [1]-[15]). Recently Bhaskar and Lakshmikantham [4], Nieto and Lopez [11]-[12], Ran and Reurings [13] and Agarwal, El-Gebeily and O'Regan [2] presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [4] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem.

Recently, Luong and Thuan [10] presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space which are generalizations of the results of Bhaskar and Lakshmikantham [4]. In this paper, we establish the existence and uniqueness of coupled coincidence point involving a (ϕ, ψ)-contractive condition for mappings having the mixed g-monotone property. We also illustrate our results with the help of an example.

2 Preliminaries

A partial order is a binary relation $\preceq$ over a set X which is reflexive, antisymmetric, and transitive. Now, let us recall the definition of the monotonic function $f : X \to X$ in the partially order set $(X, \preceq)$. We say that f is non-decreasing if for $x, y \in X$, $x \preceq y$, we have $fx \preceq fy$. Similarly, we say that f is non-increasing if for $x, y \in X$, $x \preceq y$, we have $fx \succeq fy$. Any one could read on [8] for more details on fixed point theory

Definition 2.1 *[9](Mixed g-Monotone Property)*
Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that the mapping F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument, F is monotone g-non-increasing in its second argument and F is monotone g-non-decreasing in its third argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z) \tag{1}$$

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z), \tag{2}$$

and

$$z_1, z_2 \in X, gz_1 \preceq gz_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2) \tag{3}$$

Definition 2.2 *[9] (Tripled Coincidence Point)*
Let $(x, y, z) \in X \times X \times X$, $F : X \times X \times X \to X$ and $g : X \to X$. We say that (x, y, z) is a tripled coincidence point of F and g if $F(x, y, z) = gx$, $F(y, x, z) = gy$ and $F(z, y, x) = gz$ for $x, y, z \in X$.

Definition 2.3 *[9] Let X be a non-empty set and let $F : X \times X \times X \to X$ and $g : X \to X$. We say F and g are commutative if, for all $x, y, z \in X$,*

$$g(F(x, y, z)) = F(gx, gy, gz).$$

Definition 2.4 *[5] The mapping F and g where $F : X \times X \times X \to X$ and $g : X \to X$, are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) = 0,$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, z_n)), F(gy_n, gx_n, gz_n)) = 0,$$

and

$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) = 0$$

whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n = x$, $\lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} gy_n = y$, and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n = z$ for all $x, y, z \in X$ are satisfied.

3 Existence of Tripled Coincidence Points

As in [10], let ϕ denote all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy

(i) ϕ is continuous and non-decreasing,

(ii) $\phi(t) = 0$ if and only if $t = 0$,

(iii) $\phi(t + s) \leq \phi(t) + \phi(s),\ \forall\ t, s \in [0, \infty)$.

and let ψ denote all function $\psi : [0, \infty) \to (0, \infty)$ which satisfy $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

For example [10], functions $\phi_1(t) = kt$ where $k > 0, \phi_2(t) = \frac{t}{t+1}, \phi_3(t) = \ln(t + 1)$, and $\phi_4(t) = \min\{t, 1\}$ are in $\Phi; \psi_1(t) = kt$ where $k > 0, \psi_2(t) = \frac{\ln(2t+1)}{2}$, and

$$\psi_3(t) = \begin{cases} 1, & t = 0 \\ \frac{t}{t+1}, & 0 < t < 1 \\ 1, & t = 1 \\ \frac{1}{2}t, & t > 1 \end{cases}$$

are in Ψ,

Now let us start proving our main results.

Theorem 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed g-monotone property on X such that there exist three elements $x_0, y_0, z_0 \in X$ with*

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, z_0) \text{and} \quad gz_0 \preceq F(z_0, y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned} \phi(d(F(x, y, z), F(u, v, w))) \ &\leq \ \frac{1}{3}\phi(d(gx, gu) + d(gy, gv) + d(gz, gw)) \\ &- \ \psi\left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gw)}{3}\right) \end{aligned} \quad (4)$$

for all $x, y, z, u, v, w \in X$ with $gx \succeq gu$, $gy \preceq gv$ and $gz \succeq gw$. Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous and compatible with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all n,

(iii) if a non-decreasing sequence $\{z_n\} \to z$, then $z_n \preceq z$, for all n,

then there exist $x, y, z \in X$ such that

$$gx = F(x, y, z) \quad gy = F(y, x, z) \quad \text{and} \quad gz = F(z, y, x)$$

that is, F and g have a tripled coincidence point in X.

Proof. Let $x_0, y_0, z_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0, z_0)$ and $gy_0 \succeq F(y_0, x_0, z_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$. Using $F(X \times X \times X) \subseteq g(X)$, we construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X as, for all $n \geq 0$,

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, z_n) \quad \text{and} \quad gz_{n+1} = F(z_n, y_n, x_n). \tag{5}$$

We are going to prove that, for all $n \geq 0$,

$$gx_n \preceq gx_{n+1}, \tag{6}$$

$$gy_n \succeq gy_{n+1} \tag{7}$$

and

$$gz_n \preceq gz_{n+1}. \tag{8}$$

To prove these, we are going to use the mathematical induction.

Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, z_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$ and since $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, z_0)$ and $gz_1 = F(z_0, y_0, x_0)$, we have $gx_0 \preceq gx_1$, $gy_0 \succeq gy_1$ and $gz_0 \preceq gz_1$. Thus (6), (7) and (8) hold for $n = 0$.

Suppose now that (6), (7) and (8) hold for some fixed $n \geq 0$, Then, since $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1}$ and by mixed g-monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \succeq F(x_n, y_{n+1}, z_{n+1}) \succeq F(x_n, y_n, z_n) = gx_{n+1}, \tag{9}$$

$$gy_{n+2} = F(y_{n+1}, x_{n+1}, z_{n+1}) \preceq F(y_n, x_{n+1}, z_{n+1}) \preceq F(y_n, x_n, z_n) = gy_{n+1} \tag{10}$$

and

$$gz_{n+2} = F(z_{n+1}, y_{n+1}, x_{n+1}) \succeq F(z_n, y_{n+1}, x_{n+1}) \succeq F(z_n, y_n, x_n) = gz_{n+1}. \tag{11}$$

Using (9), (10) and (11), we get

$$gx_{n+1} \preceq gx_{n+2}, \quad gy_{n+1} \succeq gy_{n+2} \quad \text{and} \quad gz_{n+1} \preceq gz_{n+2}.$$

Hence by the mathematical induction we conclude that (6), (7) and (8) hold for all $n \geq 0$. Therefore,

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots, \tag{12}$$

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots \tag{13}$$

and

$$gz_0 \preceq gz_1 \preceq gz_2 \preceq \cdots \preceq gz_n \preceq gz_{n+1} \preceq \cdots. \tag{14}$$

Since $gx_n \succeq gx_{n-1}$, $gy_n \preceq gy_{n-1}$, $gz_n \succeq gz_{n-1}$ and using (4) and (5), we have

$$\begin{aligned} \phi(d(gx_{n+1}, gx_n)) &= \phi(d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))) \\ &\leq \frac{1}{3}\phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\ &- \psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3}\right). \end{aligned} \tag{15}$$

Similarly, since $gy_{n-1} \succeq gy_n$, $gx_{n-1} \preceq gx_n$, $gz_n \succeq gz_{n-1}$ and using (4) and (5), we also have

$$\begin{aligned} \phi(d(gy_n, gy_{n+1})) &= \phi(d(F(y_{n-1}, x_{n-1}, z_{n-1}), F(y_n, x_n, z_n))) \\ &\leq \frac{1}{3}\phi(d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) + d(gz_n, gz_{n-1})) \\ &- \psi\left(\frac{d((gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) + d(gz_n, gz_{n-1})}{3}\right). \end{aligned} \tag{16}$$

Similarly, since $gz_n \succeq gz_{n-1}$, $gy_{n-1} \succeq gy_n$, $gx_{n-1} \preceq gx_n$, and using (4) and (5), we also have

$$\begin{aligned} \phi(d(gz_{n+1}, gz_n)) &= \phi(d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}))) \\ &\leq \frac{1}{3}\phi(d(gz_n, gz_{n-1}) + d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})) \\ &- \psi\left(\frac{d(gz_n, gz_{n-1}) + d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})}{3}\right). \end{aligned} \tag{17}$$

Using (15), (16) and (17), we have

$$\begin{aligned} & \phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)) + \phi(d(gz_{n+1}, gz_n)) \\ \leq\ & \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\ -\ & 3\psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3}\right). \end{aligned} \tag{18}$$

By property (iii) of ϕ, we have

$$\begin{aligned} & \phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) \\ \leq\ & \phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)) + \phi(d(gz_{n+1}, gz_n)). \end{aligned} \tag{19}$$

Using (18) and (19), we have

$$\begin{aligned} & \phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) \\ \leq\ & \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\ -\ & 3\psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3}\right). \end{aligned} \tag{20}$$

which implies, since ψ is a non-negative function,

$$\begin{aligned} & \phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) \\ \leq\ & \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})). \end{aligned}$$

Using the fact that ϕ is non-decreasing, we get

$$\begin{aligned} & d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) \\ \leq\ & d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1}). \end{aligned}$$

Set

$$\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n).$$

Now we would like to show that $\delta_n \to 0$ as $n \to \infty$. It is clear that the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)] = \delta. \tag{21}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (20) and remembering $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\begin{aligned} \phi(\delta) &= \lim_{n\to\infty} \phi(\delta_n) \leq \lim_{n\to\infty} \left[\phi(\delta_{n-1}) - 3\psi\left(\frac{\delta_{n-1}}{3}\right)\right] \\ &= \phi(\delta) - 3 \lim_{\delta_{n-1}\to\delta} \psi\left(\frac{\delta_{n-1}}{3}\right) < \phi(\delta) \end{aligned}$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)] = 0. \tag{22}$$

Now, we will prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$, $\{gy_n\}$ or $\{gz_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}, \{gx_{m(k)}\}$ of $\{gx_n\}$, $\{gy_{n(k)}\}, \{gy_{m(k)}\}$ of $\{gy_n\}$ and $\{gz_{n(k)}\}, \{gz_{m(k)}\}$ of $\{gz_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \geq \epsilon. \tag{23}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (23). Then

$$d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) + d(gz_{n(k)-1}, gz_{m(k)}) < \epsilon. \tag{24}$$

Using (23), (24) and the triangle inequality, we have

$$\begin{aligned}
\epsilon &\le r_k := d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + +d(gz_{n(k)}, gz_{m(k)}) \\
&\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) \\
&+ d(gz_{n(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{m(k)}) \\
&\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gz_{n(k)}, gz_{n(k)-1}) + \epsilon.
\end{aligned}$$

Letting $k \to \infty$ and using (22), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} [d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)} + d(gz_{n(k)}, gz_{m(k)})] = \epsilon. \tag{25}$$

By the triangle inequality

$$\begin{aligned}
r_k &= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \\
&\le d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}) \\
&+ d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}) \\
&+ d(gz_{n(k)}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{m(k)}) \\
&= \delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) \\
&+ d(gz_{n(k)+1}, gz_{m(k)+1}).
\end{aligned}$$

Using the property of ϕ, we have

$$\begin{aligned}
\phi(r_k) &= \phi(\delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) \\
&+ d(gz_{n(k)+1}, gz_{m(k)+1})) \\
&\le \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(d(gx_{n(k)+1}, gx_{m(k)+1})) \\
&+ \phi(d(gy_{n(k)+1}, gy_{m(k)+1})) + \phi(d(gz_{n(k)+1}, gz_{m(k)+1})).
\end{aligned} \tag{26}$$

Since $n(k) > m(k)$, hence $gx_{n(k)} \succeq gx_{m(k)}$, $gy_{n(k)} \preceq gy_{m(k)}$ and $gz_{n(k)} \succeq gz_{m(k)}$. Using (4) and (5), we get

$$\begin{aligned}
&\phi(d(gx_{n(k)+1}, gx_{m(k)+1})) \\
&= \phi(d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{n(k)}))) \\
&\le \frac{1}{3}\phi(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)})) \\
&- \psi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)})}{3}\right) \\
&= \frac{1}{3}\phi(r_k) - \psi\left(\frac{r_k}{3}\right).
\end{aligned} \tag{27}$$

By the same way, we also have

$$\begin{aligned}
& \phi(d(gy_{m(k)+1}, gy_{n(k)+1})) \\
= \;& \phi(d(F(y_{m(k)}, x_{m(k)}, z_{n(k)}), F(y_{n(k)}, x_{n(k)}, z_{n(k)}))) \\
\leq \;& \frac{1}{3}\phi(d(gy_{m(k)}, gy_{n(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gz_{n(k)}, gz_{m(k)})) \\
- \;& \psi\left(\frac{d(gy_{m(k)}, gy_{n(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gz_{n(k)}, gz_{m(k)})}{3}\right) \\
= \;& \frac{1}{3}\phi(r_k) - \psi(\frac{r_k}{3}). \qquad (28)
\end{aligned}$$

Also by the same way, we also have

$$\begin{aligned}
& \phi(d(gz_{n(k)+1}, gz_{m(k)+1})) \\
= \;& \phi(d(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{n(k)}))) \\
\leq \;& \frac{1}{3}\phi(d(gz_{n(k)}, gz_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gx_{n(k)}, gx_{m(k)})) \\
- \;& \psi\left(\frac{d(gz_{n(k)}, gz_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gx_{n(k)}, gx_{m(k)})}{3}\right) \\
= \;& \frac{1}{3}\phi(r_k) - \psi\left(\frac{r_k}{3}\right). \qquad (29)
\end{aligned}$$

Inserting (27), (28) and (29) in (26), we have

$$\phi(r_k) \leq \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(r_k) - 3\psi(\frac{r_k}{3}).$$

Letting $k \to \infty$ and using (22) and (25), we get

$$\phi(\epsilon) \leq \phi(0) + \phi(\epsilon) - 3\lim_{k\to\infty}\psi(\frac{r_k}{3}) = \phi(\epsilon) - 3\lim_{r_k\to\epsilon}\psi(\frac{r_k}{3}) < \phi(\epsilon)$$

a contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n = x, \quad \lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} gy_n = y,$$

$$\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n = z. \qquad (30)$$

Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) = 0, \qquad (31)$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, z_n)), F(gy_n, gx_n, gz_n)) = 0 \qquad (32)$$

and

$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) = 0. \tag{33}$$

We now show that $gx = F(x, y, z)$, $gy = F(y, x, z)$ and $gz = F(z, y, x)$. Suppose that the assumption (a) holds. For all $n \geq 0$, we have ,

$$d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n).$$

Taking the limit as $n \to \infty$, using (5), (30), (31) and the fact that F and g are continuous, we have $d(gx, F(x, y, z)) = 0$. Similarly, using (5), (30), (32) and the fact that F and g are continuous, we have $d(gy, F(y, x, z)) = 0$. Similarly, using (5), (30), (33) and the fact that F and g are continuous, we have $d(gz, F(z, y, x)) = 0$. From all of theses, we get

$$gx = F(x, y, z) \quad gy = F(y, x, z) \quad \text{and} \quad gz = F(z, y, x).$$

Finally, suppose that (b) holds. By (6), (7) and (30), we have $\{gx_n\}$ is a non-decreasing sequence, $gx_n \to x$, $\{gy_n\}$ is a non-increasing sequence, $gy_n \to y$ and $\{gz_n\}$ is a non-decreasing sequence, $gz_n \to z$,as $n \to \infty$. Hence, by assumption (b), we have for all $n \geq 0$,

$$gx_n \preceq x, \quad gy_n \succeq y \quad \text{and} \quad gz_n \preceq z. \tag{34}$$

Since F and g are compatible mappings and g is continuous, by (31), (32) and (33) we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g(F(x_n, y_n, z_n)) = \lim_{n\to\infty} F(gx_n, gy_n, gz_n), \tag{35}$$

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g(F(y_n, x_n, z_n)) = \lim_{n\to\infty} F(gy_n, gx_n, gz_n). \tag{36}$$

and

$$\lim_{n\to\infty} g(gz_n) = gz = \lim_{n\to\infty} g(F(z_n, y_n, x_n)) = \lim_{n\to\infty} F(gz_n, gy_n, gx_n). \tag{37}$$

Now we have

$$d(gx, F(x, y, z)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y, z)).$$

Taking $n \to \infty$ in the above inequality, using (5) and (26) we have,

$$\begin{aligned} d(gx, F(x, y, z)) &\leq \lim_{n\to\infty} d(gx, g(gx_{n+1})) + \lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(x, y, z)) \\ &\leq \lim_{n\to\infty} d(F(gx_n, gy_n, gz_n)), F(x, y, z)). \end{aligned} \tag{38}$$

Using the property of ϕ, we get

$$\phi(d(gx, F(x, y, z))) \leq \lim_{n\to\infty} \phi(d(F(gx_n, gy_n, gz_n)), F(x, y, z)))$$

Since the mapping g is monotone increasing, using (4), (34) and (38), we have for all $n \geq 0$,

$$\begin{aligned}\phi(d(gx, F(x,y,z))) &\leq \lim_{n\to\infty} \frac{1}{3}\phi(d(ggx_n, gx) + d(ggy_n, gy) + d(ggz_n, gz)) \\ &- \lim_{n\to\infty} \psi\left(\frac{d(ggx_n, gx) + d(ggy_n, gy) + d(ggz_n, gz)}{3}\right).\end{aligned}$$

Using the above inequality, (30) and the property of ψ, we get $\phi(d(gx, F(x,y,z))) = 0$, thus $d(gx, F(x,y,z)) = 0$. Hence $gx = F(x,y,z)$.

Similarly, we can show that $gy = F(y,x,z)$ and $gz = F(z,y,x)$. Thus we proved that F and g have a tripled coincidence point.

4 Uniqueness of Tripled Coincidence Point

In this section, we will prove the uniqueness of the tripled coincidence point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X \times X$ with the following partial order relation, for all $(x,y,z), (u,v,w) \in X \times X \times X$,

$$(x,y,z) \preceq (u,v,w) \quad \Leftrightarrow \quad x \preceq u, y \succeq v, z \preceq w.$$

Theorem 4.1 *In addition to hypotheses of Theorem 3.1, suppose that for every* $(x,y,z), (x_1,y_1,z_1)$ *in* $X \times X \times X$ *there exists a* (u,v,w) *in* $X \times X \times X$ *that is comparable to* (x,y,z) *and* (x_1,y_1,z_1)*, then* F *and* g *have a unique tripled coincidence point.*

Proof. From Theorem 3.1, the set of tripled coincidence points of F and g is non-empty. Suppose (x,y,z) and (x_1,y_1,z_1) are tripled coincidence points of F and g, that is $gx = F(x,y,z), gy = F(y,x,z), gz = F(z,y,x), gx_1 = F(x_1,y_1,z_1), gy_1 = F(y_1,x_1,z_1)$ ans $gz_1 = F(z_1,y_1,x_1)$. We are going to show that $gx = gx_1, gy = gy_1$ and $gz = gz_1$. By assumption, there exists $(u,v,w) \subset X \times X \times X$ that is comparable to (x,y,z) and (x_1,y_1,z_1). We define sequences $\{gu_n\}, \{gv_n\}$ and $\{gw_n\}$ as follows

$$u_0 = u, \quad v_0 = v, \quad w_0 = w, \quad gu_{u+1} = F(u_n, v_n, w_n), \quad gv_{n+1} = F(v_n, u_n, w_n)$$

$$\text{and} \quad gw_{n+1} = F(w_n, v_n, u_n) \quad \text{for all } n.$$

Since (u,v,w) is comparable with (x,y,z), we may assume that $(x,y,z) \succeq (u,v,w) = (u_0,v_0,w_0)$. Using the mathematical induction, it is easy to prove that

$$(x,y,z) \succeq (u_n, v_n, w_n) \quad \text{for all } n. \tag{39}$$

Using (4) and (39), we have

$$\begin{aligned}\varphi(d(gx, gu_{n+1})) &= \varphi(d(F(x, y, z), F(u_n, v_n, w_n))) \\ &< \frac{1}{3}\varphi(d(x, u_n) + d(y, v_n) + d(z, w_n)) \\ &- \psi\left(\frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}\right). \end{aligned} \tag{40}$$

Similarly

$$\begin{aligned}\varphi(d(gv_{n+1}, gy)) &= \varphi(d(F(v_n, u_n, w_n), F(y, x, z))) \\ &< \frac{1}{3}\varphi(d(v_n, y) + d(u_n, x) + d(w_n, z)) \\ &- \psi\left(\frac{d(v_n, y) + d(u_n, x) + d(w_n, z)}{3}\right), \end{aligned} \tag{41}$$

$$\begin{aligned}\varphi(d(gz, gw_{n+1})) &= \varphi(d(F(z, y, x), F(w_n, v_n, u_n))) \\ &< \frac{1}{3}\varphi(d(z, w_n) + d(y, v_n) + d(x, u_n)) \\ &- \psi\left(\frac{d(z, w_n) + d(y, v_n) + d(x, u_n)}{3}\right). \end{aligned} \tag{42}$$

Using (40), (41), (42) and the property of φ, we have

$$\begin{aligned} &\varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})) \\ \leq\ &\varphi(d(gx, gu_{n+1})) + \varphi(d(gy, gv_{n+1})) + \varphi(d(gz, gw_{n+1})) \\ \leq\ &\varphi(d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)) \\ -\ &3\psi\left(\frac{d(gx, gu_n) + d(gy, gv_n)}{3}\right). \end{aligned} \tag{43}$$

which implies, using the property of ψ,

$$\varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})) \leq \varphi(d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)).$$

Thus, using the property of ϕ,

$$d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1}) \leq d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n).$$

That is the sequence $\{d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)\}$ is decreasing. Therefore, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] = \alpha. \tag{44}$$

We will show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in (43), we have, using the property of ψ,

$$\varphi(\alpha) \leq \varphi(\alpha) - 3 \lim_{n\to\infty} \psi\left(\frac{d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)}{3}\right) < \varphi(\alpha)$$

a contradiction. Thus. $\alpha = 0$, that is,

$$\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] = 0.$$

It implies

$$\lim_{n\to\infty} d(gx, gu_n) = \lim_{n\to\infty} d(gy, gv_n) = \lim_{n\to\infty} d(gz, gw_n) = 0. \tag{45}$$

Similarly, we show that

$$\lim_{n\to\infty} d(gx_1, gu_n) = \lim_{n\to\infty} d(gy_1, gv_n) = \lim_{n\to\infty} d(gz_1, gw_n) = 0. \tag{46}$$

Using (45) and (46) we have $gx = gx_1$, $gy = gy_1$ and $gz = gz_1$.

Corollary 4.1 *[10] In addition to hypotheses of Theorem 3.1, suppose that for every $(x, y, z), (x_1, y_1, z_1)$ in $X \times X \times X$, there exists a (u, v, w) in $X \times X \times X$ that is comparable to (x, y, z) and (x_1, y_1, z_1), then F and g have a unique tripled coincidence point.*

5 Example

Example 5.1 *Let $X = R$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let*

$$d(x, y) = |x - y| \quad \textit{for} \quad x, y \in X.$$

Then (X, d) is a complete metric space.

Let $g : X \to X$ be defined as

$$gx = x^2, \ \textit{for all}\ x \in X,$$

and let $F : X \times X \times X \to X$ be defined as

$$F(x, y, z) = \frac{2x^2 - 2y^2 + 8z^2 + 1}{3}$$

F obeys the mixed g- monotone property.

It is easy to check that $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is the unique tripled coincidence point of F and g.

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