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On r-Helix Hypersurfaces

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Abstract

In this paper, we study strong r-helix hypersurfaces and the special curves on these surfaces. Moreover, we investigated the relations between strong r-helix hypersurfaces and the Gauss transformations of these surfaces in Euclidean n-space.

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1. Introduction

In differential geometry of surfaces, an helix hypersurface in E^n is defined by the property that tangent planes make a constant angle with a fixed direction (helix direction) in [3]. Di Scala and Ruiz- Hernández have introduced the concept of these surfaces in [3]. Moreover, the concept of strong r-helix submanifold of IR^n was introduced in [2]. Let $M \subset \mathrm{IR}^n$ be a submanifold and let $H(M)$ be the set of helix directions of M. If $H(M)$ is r-dimensional linear subspace of IR^n, then M is called a strong r-helix [2].

Nowadays, M. Ghomi worked out the shadow problem given by H.Wente. And, He mentioned the shadow boundary in [8]. Ruiz- Hernández investigated that shadow bounderies are related to helix submanifolds whose tangent space makes constant angle with a fixed direction in [6] .

Helix hypersurfaces has been worked in nonflat ambient spaces in [4,5]. Cermelli and Di Scala have also studied helix hypersurfaces in liquid cristals in [9].

A.I. Nistor has also introduced certain constant angle surfaces constructed on curves in E^3 in [1]. Özkaldi and Yaylı give some characterization for a curve lying on a surface for which the unit normal makes a constant angle with a fixed direction in [10].

One of the main purposes of this work is to observe the relations between strong r-helix hypersurfaces and special curves in Euclidean n-space E^n. Another purpose of this study is to give the relations between strong r-helix hypersurfaces and the Gauss transformations of these surfaces in Euclidean n-space E^n.

2. Preliminaries

Definition 2.1 Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to E^n$ be an arbitrary curve in E^n. Recall that the curve α is said to be of unit speed (or parametrized by the arc-length function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle , \rangle$ is the standart scalar product in the Euclidean space E^n given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i ,$$

for each $X = (x_1, x_2, ..., x_n), Y = (y_1, y_2, ..., y_n) \in E^n$.

Let $\{V_1(s), V_2(s), ..., V_n(s)\}$ be the moving frame along α, where the vectors V_i are mutually orthogonal vectors satisfying $\langle V_i, V_i \rangle = 1$. The Frenet equations for α are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix}.$$

Recall that the functions $k_i(s)$ are called the i-th curvatures of α [7].

Definition 2.2 Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to E^n$ be a unit speed curve with nonzero curvatures k_i $(i = 1,2,...,n)$ in E^n and let $\{V_1, V_2, ...V_n\}$ denote the Frenet frame of the curve α. We call α a V_n-slant helix, if the n-th unit vector field V_n makes a constant angle φ with a fixed direction X, that is,

$$\langle V_n, \mathrm{X} \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant}$$

along the curve, where X is unit vector field in E^n [7].

Definition 2.3 Given a hypersurface $M \subset \mathrm{IR}^n$ and an unitary vector $d \neq 0$ in IR^n, we say that M is a helix with respect to the fixed direction d if for each $q \in M$ the angle between d and $T_q M$ is constant. Note that the above definition is equivalent to the fact that $\langle d, \xi \rangle$ is constant function along M, where ξ is a unit normal vector field on M [3].

Definition 2.4 A submanifold $M \subset \mathrm{IR}^n$ is a $r-$helix if there is exist a linear subspace $H \subset \mathrm{IR}^n$ of dimension $r = \dim(H)$ such that M is a helix with respect to any direction $d \in H$. The subspace H is called the subspace of helix directions [3].

Definition 2.5 Let $M \subset \mathrm{IR}^n$ be a submanifold of a euclidean space. A vector $d \in \mathrm{IR}^n$ is called a helix direction of M if the angle between d and any tangent space $T_p M$ is constant. Let $H(M)$ be the set of helix directions of M. We say that M is a strong r-helix if $H(M)$ is r-dimensional linear subspace of IR^n [2].

Theorem 2.1 Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to \mathrm{E}^n$ be a unit speed curve (parametrized by arclength function s) in E^n and let $\beta : \mathrm{I} \to \mathrm{S}^{n-1} \subset \mathrm{E}^n$ be the tangent indicatrix of the curve α, where S^{n-1} is the unit hypersphere in E^n. Then the curve α is a slant helix with direction L in E^n if and only if the curve β is a general helix (spherical helix) with direction L on $S^{n-1} \subset E^n$. In other words, α and β have the same direction L [11].

Theorem 2.2 Let M be a $r-$helix hypersurface in E^n and let H be the subspace of helix directions of M. If $\alpha : I \subset IR \to M$ be a unit speed geodesic curve on M, then the curve α is a V_2-slant helix with respect to any direction $d \in H$ in E^n [12].

Proof: Let ξ be a normal vector field on M. Since M is $r-$helix hypersurface, $<d,\xi>=$ constant, where $d \in H$ is any direction. That is, the angle between d and ξ is constant on every point of the surface M. And, $\alpha''(s) = \lambda\xi \mid_{\alpha(s)}$ along the curve α since α is a geodesic curve on M. Moreover, by using the Frenet equation $\alpha''(s) = V_1' = k_1 V_2$, we obtain $\lambda\xi \mid_{\alpha(s)} = k_1 V_2$, where k_1 is first curvature of α. Thus, from the last equation, by taking norms on both sides, we obtain $\xi = V_2$ or $\xi = -V_2$. So, $\langle d, V_2 \rangle$ is constant along the curve α since $\langle d, \xi \rangle =$ constant. In other words, the angle between d and V_2 is constant along the curve α. Consequently, the curve α is a V_2-slant helix with respect to any direction $d \in H$ in E^n.

3. MAIN THEOREMS

Theorem 3.1 Let $\alpha : I \subset IR \to M \subset E^n$ be a geodesic curve on M with unit speed (or parametrized by arclength function s) and let $H \subset IR^n$ be the subspace of helix directions of M, where M is a r-helix hypersurface. Then the tangent indicatrix $\alpha'(s)$ of the curve $\alpha(s)$ is a spherical helix (general helix) with respect to any helix direction in H on unit the hypersphere $S^{n-1} \subset E^n$.

Proof: We assume that α is a geodesic curve on M. Then from Theorem 2.2 the curve α is a V_2-slant helix with respect to any helix direction $d \in H$ in E^n. On the other hand, from Theorem 2.1, the tangent indicatrix α' of the curve α with the direction d is a spherical helix with respect to the same direction d in H on the unit hypersphere $S^{n-1} \subset E^n$. Consequently, α' is a spherical helix (general helix) with respect to any helix direction in H on the unit hypersphere $S^{n-1} \subset E^n$. This completes the proof.

Theorem 3.2 Let M be a hypersurface in Euclidean n-space E^n and let N be the unit normal vector field of M. If M is a strong r-helix in E^n, where $H(M) \subset IR^n$ is the space of helix directions of M, then the position vectors on all points of the surface

$$\eta(M)=\left\{X\in \mathrm{S}^{n-1} \mid X=N(P), P\in M\right\}$$

make a constant angle with the space $H(M)\subset \mathrm{IR}^n$. Here,

$$\eta : M \to \mathrm{S}^{n-1} \subset \mathrm{E}^n$$
$$P \to \eta(P)=\vec{N}(P)$$

is Gauss transformation of M, where S^{n-1} is the unit hypersphere in E^n.

Proof: Let $d\in H(M)$ be any helix direction of M. Since M is a strong r-helix in E^n,

$$< N(P), d >= \text{constant}$$

for every $P\in M$. So, we obtain

$$< X, d >= \text{constant},$$

for every $X=\vec{N}(P)\in \eta(M)$, where $d\in H(M)$ any helix direction of M. It follows that

$$< X, d >= \text{constant}$$

for all $d\in H(M)$, where X is the position vector of $\mu(M)$. Hence, the position vectors on all points of the surface $\mu(M)$ make a constant angle with the space $H(M)\subset \mathrm{IR}^n$. This completes the proof.

Theorem 3.3 Let M be a strong r-helix hypersurface in Euclidean n-space E^n and let $H(M)\subset \mathrm{IR}^n$ be the space of helix directions of M. If $\alpha : \mathrm{I}\subset \mathrm{IR}\to M$ $(\alpha(t)\in \mathrm{I}, t\in \mathrm{I})$ is a curve on the surface M and $\exists d_j \notin T_pM$ for every $p=\alpha(t)\in M$, where $d_j\in H(M)$, then the curve

$$\eta(\alpha(t))=\left\{X\in S^{n-1} \mid X=N|_{\alpha(t)}, \alpha(t)\in M\right\}$$

is not a geodesic curve on the unit hypersphere $S^{n-1}\subset \mathrm{E}^n$, where $N|_{\alpha(t)}$ the unit normal vector field of M along the curve α and

$$\eta : M \to S^{n-1} \subset \mathrm{E}^{\mathrm{n}}$$
$$\alpha(t) \to \eta(\alpha(t))=\vec{N}|_{\alpha(t)}$$

Gauss transformation of M on the curve α.

Proof: Since M is a strong r-helix hypersurface, for all $d_j\in H(M)$,

$$< N|_{\alpha(t)}, d_j >= \cos(\theta_j) = \text{constant}$$

along the curve α. We assume that $\beta = \eta(\alpha(t))$, where η Gauss transformation of M. So, we obtain

$$\beta = N|_{\alpha(t)}$$

along the curve α since $\eta(\alpha(t)) = N|_{\alpha(t)}$ along the curve. Thus, we have

$$< \beta, d_j >= \cos(\theta_j) = \text{constant}$$

for all $d_j \in H(M)$.

If we take the derivative in each part of the equality $< \beta, d_j >= \cos(\theta_j)$ twice, we obtain, for all $d_j \in H(M)$,

$$< \beta'', d_j >= 0. \tag{1}$$

Now, we suppose that the curve β is a geodesic curve on S^{n-1}. Then $\beta'' = kN|_{\alpha(t)}$, where $N|_{\alpha(t)}$ is also the normal vector field of S^{n-1}. Hence, we get, for all $d_j \in H(M)$,

$$< N|_{\alpha(t)}, d_j >= 0$$

by using the equation (1). It follows that all $d_j \in T_{\alpha(t)}M$ for every point $\alpha(t) \in M$. But, according to the hypothesis in this Theorem, $\exists d_j \notin T_pM$ for every $p = \alpha(t) \in M$. So, it is a contradiction. As a result, $\beta = \eta(\alpha(t))$ is not a geodesic curve on the unit hypersphere $S^{n-1} \subset \mathrm{E}^n$.

Theorem 3.4 Let M be strong $r-$ helix hypersurface in E^n and $H(M) \subset \mathrm{IR}^n$ be the space of helix directions of M. If $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(t) \in M$, $t \in \mathrm{I}$) is a curve on the surface M, then $d \in Sp\{\beta'\}^{\perp}$ along the curve β for all $d \in H(M)$. Here,

$$\beta = \eta(\alpha(t)) = \{X \in S^{n-1} \mid X = N|_{\alpha(t)}, \alpha(t) \in M\}$$

and

$$\begin{aligned} \eta : M &\to S^{n-1} \subset \mathrm{E}^n \\ \alpha(t) &\to \eta(\alpha(t)) = \vec{N}|_{\alpha(t)} \end{aligned}$$

is Gauss transformation of M on the curve α, where S^{n-1} is the unit hypersphere in E^n and $N|_{\alpha(t)}$ the unit normal vector field of M along the curve α.

Proof: Since M is a strong $r-$helix hypersurface in E^n,

$$< N|_{\alpha(t)}, d >= \text{constant}$$

along the curve α for all $d \in H(M)$. If we take the derivative in each part of the equality $< N|_{\alpha(t)}, d >= \text{constant}$, we have

$$< N'|_{\alpha(t)}, d >= 0 .$$

On the other hand, $\beta' = N'|_{\alpha(t)}$ since $\beta = N|_{\alpha(t)}$. So, we obtain:

$$< \beta', d >= 0$$

throughout the curve β. As a result, $d \in Sp\{\beta'\}^{\perp}$ along the curve β for all $d \in H(M)$.

Corollary 3.1 Let M be helix hypersurface with the fixed direction d in E^n. If $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(t) \in M$, $t \in \mathrm{I}$) is a curve on the surface M, then $d \in Sp\{\beta'\}^{\perp}$ along the curve β. Here,

$$\beta = \eta(\alpha(t)) = \{X \in S^{n-1} \mid X = N|_{\alpha(t)}, \alpha(t) \in M\}$$

and

$$\begin{aligned} \eta : M &\to S^{n-1} \subset \mathrm{E}^n \\ \alpha(t) &\to \eta(\alpha(t)) = \vec{N}|_{\alpha(t)} \end{aligned}$$

is Gauss transformation of M on the curve α, where S^{n-1} is the unit hypersphere in E^n and $N|_{\alpha(t)}$ the unit normal vector field of M along the curve α.

Theorem 3.5 Let M be strong $r-$helix hypersurface in E^n and let $H(M)$ be the space of helix directions of M. If $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(s) \in M$, $s \in \mathrm{I}$) is a curve with unit speed on the surface M (s arc-length parameter) and if α is a geodesic curve on M, then $d \in Sp\{V_2'\}^{\perp}$ along the curve α, where V_2 is the element of the moving frame $\{V_1(s), V_2(s), ..., V_n(s)\}$ of the curve α and $d \in H(M)$ is any direction.

Proof: Since M is strong $r-$ helix hypersurface, for any direction $d \in H(M)$,

$$< N\,|_{\alpha(s)}, d >= \text{constant}$$

along the curve α, where N is the normal vector field of M. If we are taking the derivative in each part of the equality with respect to s, we obtain :

$$\left\langle (N\,|_{\alpha(s)})', d \right\rangle = 0 \tag{2}$$

along the curve α. And, $\alpha''(s) = \lambda N\,|_{\alpha(s)}$ along the curve α since α is a geodesic curve on M. Moreover, by using the Frenet equation $\alpha''(s) = V_1' = k_1 V_2$, we obtain $\lambda N\,|_{\alpha(s)} = k_1 V_2$, where k_1 is first curvature of α. Thus, from the last equation, by taking norms on both sides, we obtain $N\,|_{\alpha(s)} = V_2$ or $N\,|_{\alpha(s)} = -V_2$. So, $(N\,|_{\alpha(s)})' = V_2'$. It follows that $< V_2', d >= 0$ by using the equality (2). Consequently, $d \in Sp\{V_2'\}^{\perp}$ along the curve α.

Corollary 3.2 Let M be a helix hypersurface with the direction d in E^n and let $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(s) \in M$, $s \in \mathrm{I}$) be a curve with unit speed on the surface M (s arc-length parameter). If α is a geodesic curve on M, then $d \in Sp\{V_2'\}^{\perp}$ along the curve α, where V_2 is the element of the moving frame $\{V_1(s), V_2(s), ..., V_n(s)\}$ of the curve α.

Theorem 3.6 Let M be a strong r-helix hypersurface in E^n and let $H(M) \subset \mathrm{IR}^n$ be the space of the helix directions of M. Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(s) \in M$, $s \in \mathrm{I}$) be a curve with unit speed on the surface M (s arc-length parameter). Let us assume that for each $q \in M$ the angle $\theta_j \neq \{0, \pi/2\}$ between each $d_j \in H(M)$ and N. If for $\exists d_j \in H(M)$ $d_j \in Sp\{V_1, N\}$ along the curve α, then the curve α is a asymptotic curve on the surface M, where V_1 is the element of the Frenet frame $\{V_1, V_2, ..., V_n\}$ and N is the unit normal vector field of the surface M.

Proof: Since M is a strong r-helix hypersurface and $d_j \in Sp\{V_1, N\}$ for $\exists d_j \in H(M)$ along the curve α, we can write:

$$d_j = \cos(\theta_j)N + \sin(\theta_j)V_1 . \quad (3)$$

Taking the derivative in each part of the equation (3), we obtain:

$$0 = \cos(\theta_j)N' + \sin(\theta_j)V_1' \quad (4)$$

And, doing the scalar product with V_1 in each part of the equation (4), we have:

$$0 = \cos(\theta_j) < N', V_1 > + \sin(\theta_j) < V_1', V_1 > \quad (5)$$

On the other hand, $< V_1', V_1 >= 0$ since V_1 is a unit vector. So, we get from the equation (5):

$$\cos(\theta_j) < N', V_1 >= 0 .$$

According to hypothesis in this Theorem, since $\cos(\theta_j) \neq 0$, it follows that

$$< N', V_1 >= 0 .$$

Finally, the curve α is a asymptotic curve. This completes the proof.

Corollary 3.3 Let M be a strong r-helix hypersurface in E^n and let $H(M) \subset \mathrm{IR}^n$ be the space of the helix directions of M. Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(s) \in M$, $s \in \mathrm{I}$) be a geodesic curve with unit speed on the surface M (s arc-length parameter). Let us assume that for each $q \in M$ the angle $\theta_j \neq \{0, \pi/2\}$ between each $d_j \in H(M)$ and N. If for $\exists d_j \in H(M)$ $d_j \in Sp\{V_1, N\}$ along the curve α, then the curve α is a line on the surface M, where V_1 is the element of the Frenet frame $\{V_1, V_2, ..., V_n\}$ and N is the unit normal vector field of the surface M.

Proof: From Theorem 3.6, the curve α is a asymptotic curve. Moreover, according to the hypothesis in this Theorem, the curve α is a geodesic curve. As we know, if a curve is both asymptotic and geodesic, then the curve is a line only.

Corollary 3.4 Let M be a strong r-helix hypersurface in E^n and let $H(M) \subset \mathrm{IR}^n$ be the space of the helix directions of M. Let $\alpha : \mathrm{I} \subset \mathrm{IR} \to M$ ($\alpha(s) \in M$, $s \in \mathrm{I}$) be a line of curvature with unit speed on the surface M (s arc-length parameter). Let us assume that for each $q \in M$ the angle $\theta_j \neq \{0, \pi/2\}$ between each $d_j \in H(M)$ and N. If for $\exists d_j \in H(M)$ $d_j \in Sp\{V_1, N\}$ along the curve α, then the curve α is a line on the surface M, where V_1 is the element of the Frenet frame $\{V_1, V_2, ..., V_n\}$ and N is the unit normal vector field of the surface M.

Proof: From Theorem 3.6, the curve α is a asymptotic curve. Moreover, according to the hypothesis in this Theorem, the curve α is a line of curvature. As we know, if a curve is both asymptotic and line of curvature, then the curve is a line only.

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