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A Rees Convolution Product for Topological Semihypergroups

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Abstract

A semihypergroup is defined by dropping the requirement of an identity or involution from the definition of a hypergroup. Dunkl [Du73] called it a hypergroup (without involution) while Jewett [Je75] referred to it as a semiconvo. In this paper, we generalize some basic algebraic results from semigroups to semihypergroups. Among other things, we define a Rees convolution product for a topological semihypergroup S and prove that if X, Y are non-empty sets and H is a hypergroup, then with the Rees convolution product, $X \times H \times Y$ is a completely simple semihypergroup which has all its idempotent elements in its center. We also prove that in every locally compact semihypergroup, S, if B is a Borel subset of S then for any $x \in S$, the sets Bx^- and x^-B are also Borel subsets of S.

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1 Introduction

In contrast with topological semigroups, an algebraic operation is not defined on a topological semihypergroup S, rather the convolution of measures is used to induce an algebraic operation on S. We are therefore logically faced with the question: how much algebraic structure could be inherited from the algebra of measures of a topological semihypergroup? We address this question by first defining the corresponding algebraic concepts, on semihypergroups, verifying that our definitions remain consistent with the classical semigroup definitions. Among other things, we define the notion of ideals, simple and completely simple semihypegroups, as well as a Rees convolution product for topological semihypergroups. Along the line we give basics results, essential in doing harmonic analysis and Probability on semihypergroups. Typical results are:

1. Let X and Y be non-empty sets and H be a locally compact topological hypergroup, then with the Rees convolution product, $X \times H \times Y$ is a completely simple semihypergroup.

2. Let B be a Borel subset of a locally compact topological semihypergroup S. Then for any $x \in S$, the sets Bx^- and x^-B are also Borel subsets of S

Also we use simple finite-element semihypergroups to illustrate striking contrast between semihypergroups and semigroups. For instance a semigroup which is left and right simple is a group but we have an example of simple two-element semihypergroup which is not a hypergroup. We show that if a two-element semihypergroup is not commutative then it must be a semigroup.

All undefined terms used in this work in connection with topological semihypergroups can be found in Jewett [Je75] or Youmbi [Yn05].

2 Preliminaries

Let S be a locally compact Hausdorff space; $C(S)$, the space of complex continuous functions on S; $C_b(S)$ the space of bounded elements of $C(S)$; $C_0(S)$ the space of elements of $C_b(S)$ which tends to 0 at ∞; $C_c(S)$ the space of elements of $C_0(S)$ with compact support; $C_c^+(S)$ the space of nonnegative elements of $C_c(S)$; $M(S)$ denotes the set of finite regular Borel measures;$M_+(S)$ the set of non-negative measures; $M_1(S)$ denote the set of probability measures; If $\mu \in M(S)$ then $Supp(\mu) = \{x \in S : \text{if } V \text{ is any open set containing } x \text{ then } \mu(V) > 0\}$; An unspecified topology on $M_+(S)$ is the cone topology.

2.1 Definition

A nonempty locally compact Hausdorff space S will be called a semihypergroup if the following conditions are satisfied:

(SH_1) $(M(S), +, *)$ is a Banach algebra.

(SH_2) For all $x, y \in S$, $\delta_x * \delta_y$ is a probability measure with compact support.

(SH_3) The mapping $(x, y) \mapsto \delta_x * \delta_y$ of $S \times S$ into $M_1(S)$, where $S \times S$ has the product topology and $M_1(S)$ has the weak topology, is continuous.

(SH_4) The mapping$(x, y) \mapsto Supp(\delta_x * \delta_y)$ of $S \times S$ into $\mathcal{C}(S)$ is continuous, where $\mathcal{C}(S)$ is the space of compact subsets of S endowed with the Michael topology, that is the topology generated by the subbasis of all $\mathcal{C}_U(V) = \{C \in \mathcal{C}(S) : C \cap U \neq \emptyset \text{ and } C \subset V\}$ where U and V are open subsets of S.

If in addition,

SH_5 there exists $e \in S$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x \ \forall x \in S$, and

SH_6 There exists a topological involution (a homeomorphism) from S onto S such that $(x^-)^- = x \ \forall x \in S$, with $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ and $e \in Supp(\delta_x * \delta_y)$ if and only if $x = y^-$ where for any Borel set B, $\mu^-(B) = \mu(\{x^- : x \in B\})$.

$(S, *)$ will be called a hypergroup.

Remarks

(i) If $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in S$ we say that $(S, *)$ is a commutative semihypergroup.

(ii) The convolution $*$ on $M(S)$ is defined by

$$\mu * \nu(f) = \int_S f d\mu * \nu = \int_S \mu(dx) \int_S \nu(dy) \int_S f d\delta_x * \delta_y.$$

for all $f \in C_b(S)$.

2.2 Examples

1. If $(S, .)$ is a topological semigroup, where S is a locally compact Hausdorff space, then with convolution defined by $\delta_x * \delta_y = \delta_{xy}$, $(S, *)$ is a semihypergroup. Also if a semihypergroup is such that the convolution of two point masses is a point mass, then it is a semigroup.

2. Let $S = \{x, y\}$ with the discrete topology. Then S is a locally compact space we can define the convolution of point masses by

$$\delta_x * \delta_x = a\delta_x + b\delta_y$$

$$\delta_y * \delta_y = b'\delta_x + a'\delta_y$$

$$\delta_x * \delta_y = p\delta_x + p'\delta_y$$
$$\delta_y * \delta_x = q\delta_x + q'\delta_y$$

where $a, b, a', b', p, p', q, q'$ are non-negative real numbers such that $a + b = a' + b' = p + p' = q + q' = 1$ (for the convolution product of two point masses to be a probability measure) and $bb' = pp' = qq'$ (for the convolution product to be associative). Then $(S, *)$ is a semihypergroup.

We observe here that every commutative two-element semihypergroup is a semigroup. For if either b or b' is non-zero, the semihypergroup is commutative. Now let us assume that b and b' are both zero. Then, associativity of convolution implies that $pp' = qq' = 0$ so that the semihypergroup is actually a semigroup since one of p, p' is zero and one of q, q' is zero. Thus we most have $\delta_x*\delta_x = \delta_x$, $\delta_y*\delta_y = \delta_y$, $\delta_x*\delta_y = \delta_y$, $\delta_y*\delta_x = \delta_x$ which is a non commutative semigroup.

3. Let $S = \{e, a, b\}$. Let e be the identity element and let us define

$$\delta_a * \delta_a = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$$
$$\delta_b * \delta_b = \delta_a$$
$$\delta_a * \delta_b = \delta_b * \delta_a = \frac{1}{2}\delta_e + \frac{1}{2}\delta_b$$

Then $(S, *)$ is a semihypergroup and if we defined an involution by $a' = b$ and $b' = a$ we have

$$(\delta_a * \delta_a)' = \frac{1}{2}\delta_{a'} + \frac{1}{2}\delta_{b'} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_a$$

But

$$\delta_{a'} * \delta_{a'} = \delta_b * \delta_b = \delta_a \neq \frac{1}{2}\delta_a + \frac{1}{2}\delta_b.$$

Although $e \in Supp(\delta_a * \delta_b)$ this involution does not satisfy the condition $(\delta_a * \delta_b)' = \delta_{b'} * \delta_{a'}$, this semihypergroup is almost (though not) a hypergroup and it is called a regular semihypergroup [On93].

4. Let $H = \{e, x, y\}$ and let e be the identity element, the identity function is considered as the involution, and a commutative convolution is defined on H by

$$\delta_x * \delta_x = a\delta_e + b\delta_x + c\delta_y$$
$$\delta_y * \delta_y = a'\delta_e + c'\delta_x + b'\delta_y$$
$$\delta_x * \delta_y = \delta_y * \delta_x = q\delta_x + q'\delta_y$$

Then $(H, *)$ is a hypergroup provided $a + b + c = a' + b' + c' = q + q' = 1$ (for the convolution of two point masses to be a probability measure, and $a'c = aq$ (for associativity of convolution).

2.3 Definition

1. An element $e \in S$ is called a left (right) identity element of S if $\delta_e * \delta_x = \delta_x$ ($\delta_x * \delta_e = \delta_x$) for every $x \in S$. An element e is called a two sided identity of S or simply an identity of S, if it is both a left and right identity. The identity, when it exists, is unique.

2. An element $z \in S$ is called a left(right) zero element of S if $\delta_z * \delta_x = \delta_z$ ($\delta_x * \delta_z = \delta_z$) for all $x \in S$. If z is both left and right zero, we simply call it the zero of S. A semihypergroup has at most one zero.

3. An element $a \in S$ is called an idempotent element of S if $\delta_a * \delta_a = \delta_a$

Remark

The only idempotent element in a hypergroup is the identity element. For if there is an idempotent element, its point mass would be an idempotent measure and its support a singleton.

2.4 Definition

Let $(S, *)$ be a semihypergroup.

1. If $x \in S$ and A, B are subsets of S we define

$$Ax = \bigcup_{y \in A} Supp(\delta_y * \delta_x)$$

$$xA = \bigcup_{y \in A} Supp(\delta_x * \delta_y)$$

$$A * B = \bigcup_{x \in A, y \in B} Supp(\delta_x * \delta_y)$$

Remark

A closed nonempty subset F of S can be verified to be a subsemihypergroup of S if and only if $F * F \subset F$

2.5 Proposition [Je75]

Let S be a semihypergroup and $A, B, C \subset S$. Then

i. $\bar{A} * \bar{B} \subset \overline{A * B}$.

ii. If A and B are compact then $A * B$ is compact

iii. Convolution is a continuous operation on $\mathcal{C}(S)$

iv. If A and B are compact and U is an open set containing $A * B$, then there exist open sets V and W such that $A \subset V$, $B \subset W$ and $V * W \subset U$

v. $(A * B) * C = A * (B * C)$

$\mathcal{C}(S)$ with * so defined is a topological semigroup.

Remark [Je75]

1. If $\{x_\beta\}$ is a net in a hypergroup S, then the expression $x_\beta \to \infty$ means that $x_\beta \in S - A$ eventually for each compact subset A of S.

2. If $\{A_\beta\}$ is a net in $\mathcal{C}(S)$, then the expression $A_\beta \to \{\infty\}$ means that $A_\beta \subset S - A$ eventually for each compact subset A of S.

Note that $A_\beta \to \infty$ and $A_\beta \to \{\infty\}$ have different meanings.

The next proposition is stated without proof in [Je75]. For the sake completion, we give here a detailed proof.

2.6 Proposition

If H is a hypergroup and A, B, C are subsets of H, then

i. $e \in A^- * B$ if and only if $A \cap B \neq \emptyset$; also $e \in A * B^-$ if and only if $A \cap B \neq \emptyset$

ii. $(A*B) \cap C \neq \emptyset$ if and only if $B \cap (A^- * C) \neq \emptyset$ if and only if $A \cap (C * B^-) \neq \emptyset$

iii. If B is open, then $A * B$ is open and $\bar{A} * B = A * B$

iv. If A is compact and B is closed, then $A * B$ is closed.

Proof

i. Suppose $e \in A^- * B$. Then there exists $x \in A$ and $y \in B$ such that $e \in Supp(\delta_{x^-} * \delta_y)$ which implies $x = y$ (from SH_6), so $A \cap B \neq \emptyset$. Now if $A \cap B \neq \emptyset$ then there exists $x \in A \cap B$, and so $e \in Supp(\delta_{x^-} * \delta_x)$. Therefore, $e \in A^- * B$

ii. $(A*B) \cap C \neq \emptyset$ if and only if $e \in (A*B)^- * C$ if and only if $e \in B^- * (A^- * C)$ if and only if $B \cap (A^- * C) \neq \emptyset$ if and only if $e \in B * (C^- * A) = (B * C^-) * A$ if and only if $A \cap (C * B^-) \neq \emptyset$

iii. Suppose B is open. Let $a \in A$, then $x \in \{a\} * B$ if and only if $B \cap \{a^-\} * \{x\} \neq \emptyset$ (from ii above). Since the map $x \longmapsto \{a^-\} * \{x\}$ is continuous (from SH_4), the set $\mathcal{C}_B(H)$ is an open set in the Michael topology which contains $\{a^-\} * \{x\}$ (because $B \cap \{a^-\} * \{x\} \neq \emptyset$ and $\{a^-\} * \{x\} \subset H$) so its inverse image by the continuous function $x \longmapsto \{a^-\} * \{x\}$ is open, which is, $\{y \in H : \{a^-\} * \{y\} \cap B \neq \emptyset\} = \{a\} * B$. Thus $\{a\} * B$ is an open subset of H.

iv. Let (x_n) be a sequence of elements of $A * B$ converging to an element $x \in S$. Then there are sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $x_n \in \{a_n\} * \{b_n\}$ for each n. This is equivalent to $b_n \in \{a_n^-\} * \{x_n\}$ for each n (from (ii) above see also the remark (i) below). Since A is compact, the sequence (a_n) has a convergent subsequence say, (a_k) such that $b_k \in \{a_k^-\} * \{x_n\}$ for each k. Furthermore (a_k^-) and (x_k) are relatively compact (as convergent sequences). So (b_k) has a convergent subsequence converging to a point $b \in B$ (since B is closed). Now from SH_4 if $a_k \longrightarrow a \in A$ then $\{a_k^-\} * \{x_k\} \longrightarrow \{a^-\} * \{x\}$. So $b \in \{a^-\} * \{x\}$ (since $b_k \in \{a_k^-\} * \{x\}$ for all k). And again from (ii) above $b \in \{a^-\} * \{x\}$ if and only if $x \in \{a\} * \{b\} \subset A * B$. Thus $A * B$ is closed.

2.7 Definition

1. A **homomorphism of semihypergroups** is defined via measure algebra as follows: Let S and T be two semihypergroups. A mapping ϕ from S into T is called a semihypergroup homomorphism if and only if $\phi : (M_1(S), *) \longrightarrow (M_1(T), \bullet)$ is a semigroup homomorphism. That is, $\phi(\mu * \nu) = \phi(\mu) \bullet \phi(\nu)$, $\forall \mu, \nu \in M_1(S)$, such that $\phi(\delta_x)$ is a point mass in $M_1(T)$, $\forall x \in S$. If in addition ϕ is one to one and onto, it is referred to as an isomorphism.

2. **Product of semihypergroups**. Let $(S, *), (T, \bullet)$ be two semihypergroups. The set $S \times T$ with the product topology is a locally compact space, and this can be made into a semihypergroup by defining

$$\delta_{(x,y)} \circ \delta_{(s,t)} = \delta_x * \delta_s \otimes \delta_y \bullet \delta_t$$

where $(x, y), (s, t) \in S \times T$ and $\delta_{(x,y)} \circ \delta_{(s,t)}$ is a product measure on $S \times T$.

2.8 Definition

Let S be a locally compact semihypergroup. The center of S is defined by $Z(S) = \{x \in S : Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in S\}$

2.9 Remark

For a hypergroup H the center is the maximum subgroup defined by Jewett as $Z(H) = \{x \in H : \delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e\}$. To see this, suppose that $\delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e$ and let $y \in H$ be arbitrarily chosen, and assume that $a, b \in Supp(\delta_x * \delta_y)$; then since $a \in \{x\} * \{y\}$, from [Je75](lemma 4.1B) $\{x\} * \{y\} \cap \{a\} \neq \emptyset$ which is equivalent to $y \in \{x^-\} * \{a\}$; similarly, $y \in \{x^-\} * \{b\}$ which means that $\{x^-\} * \{a\} \cap \{x^-\} * \{b\} \neq \emptyset$ and this is equivalent to $\{a\} \cap \{x\} * \{x^-\} * \{b\} \neq \emptyset$ and since $\delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e$ it follows that $\{a\} \cap \{b\} \neq \emptyset$ that is $a = b$ so that $Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in H$.

Conversely suppose an element x is such that $Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in H$ then $Supp(\delta_x * \delta_{x^-})$is a singleton and since by definition it contains e we have $\delta_x * \delta_{x^-} = \delta_e$.

2.10 Example

i. Every semigroup is a semihypergroup and its center is the entire semigroup. Also every group is a hypergroup which is the maximum subgroup(equivalently the center) of itself.

ii. If H is a hypergroup, then $e \in H$ so the center of a hypergroup is nonempty. When $Z(H) = \{e\}$, the center is said to be trivial.

iii. Let $S = \{x, y\}$ with convolution defined by

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

from the definition of two-element semihypergroups above Example 2.2, S is a semihypergroup with a void center.

iv. Consider the segment $[0, 1]$ with convolution defined by

$$\delta_r * \delta_s = \frac{1}{2}\delta_{|r-s|} + \frac{1}{2}\delta_{1-|1-r-s|}$$

for all $r, s \in [0, 1]$ Zeuner [Ze89] proved that $([0, 1], *)$ is a hypergroup with a nontrivial center $\{0, 1\}$.

2.11 Definition

1. A subsemihypergroup L (R) of a semihypergroup S is called a left (right) ideal of S if $S * L \subset L$ ($R * S \subset R$); I is called an ideal of S if and only if it is both a right and left ideal.

2. S is called left (right) simple if it contains no proper left (right) ideal. S is said to be simple if it contains no proper ideal. A left (right) ideal is said to be a principal left (right) ideal if it is of the form $\{a\} \cup Sa$ ($\{a\} \cup aS$)for some $a \in S$ (Recall that we write Sa to mean $S * \{a\}$).

3. $\forall a, b \in S$ we say that the equation $xa = b$ is **solvable** if and only if there exists $x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$

2.12 Proposition

A semihypergroup S is left (right) simple if and only if $\forall a, b \in S$ the equation $xa = b$ ($ax = b$) is solvable.

Proof:

First, assume S is left simple. Then $\forall a \in S$, Sa is a left ideal of S and since S is left simple $S = Sa$ and it follows that $\forall b \in S$, $\exists x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$ so $xa = b$ is solvable. Now assume that $xa = b$ is solvable for all $a, b \in S$, and L is a left ideal of S. Then given $a \in L, Sa \subset L$. Also given $b \in S$ the equation $xa = b$ is solvable so $\exists x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$ which is a subset of Sa, so $S \subset Sa \subset L$ therefore $S = L$ and so S is left simple. We can also make a similar argument for right ideals.

2.13 Remark

i. Every left (right) ideal contains a left (right) ideal of the form Sa (aS) for some $a \in S$. For if L is a left ideal then for any $a \in L$, Sa is a left ideal contained in L. A similar statement holds for right ideals.

ii. A semihypergroup can be left and right simple without being a hypergroup. An example is the following semihypergroup. Let $S = \{x, y\}$ with convolution defined by

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

From example 2.10(iii) S so defined is a semihypergroup with no proper ideal but is not a hypergroup since it has no identity element.

2.14 Definition

1. An idempotent element in a semihypergroup S is said to be **a primitive idempotent element** if it is in the center of the semihypergroup and is minimal with respect to the partial order $\leq$ on $E(S)$ (the set of idempotent elements of S), defined by

$$e \leq f \Longleftrightarrow \delta_e * \delta_f = \delta_f * \delta_e = \delta_e$$

2. **A completely simple semihypergroup** is a simple semihypergroup which contains a primitive idempotent element.

2.15 Remark

The order defined on $E(S)$ uses convolution of point masses to compare idempotent elements of S. Note that if a is a primitive idempotent of S, δ_a is not necessarily a primitive idempotent in $M_1(S)$, according to the definition of primitive idempotents in the semigroup (with respect to convolution)$M_1(S)$.

With this definition every completely simple semigroup is a completely simple semihypergroup. In the semigroup theory completely simple semigroups are characterized by a product called the Rees Product. We introduce here a similar product for semihypergroups which we call a Rees Convolution Product.

3 Rees Convolution Product

Let $(H, *)$ be a hypergroup with center Z and X,Y be two nonempty sets. Let $\phi : Y \times X \longrightarrow Z$ be a mapping . Let us define a convolution on point masses of $X \times H \times Y$ by

$$\delta_{(x,h,y)} \bullet \delta_{(x',h',y')} = \delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}$$

This product will be referred to as the Rees convolution product.

3.1 Proposition

If H is a hypergroup, and X and Y are two nonempty locally compact Hausdorff spaces, then the space $X \times H \times Y$ is a semihypergroup with the Rees convolution product, as defined above.

Proof:

Let $K = X \times H \times Y$ and $(x,h,y),(x',h',y')$ be two points in K. Then

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}](K) =$$

$$[\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}](K) =$$

$$\delta_x(X)[\delta_h * \delta_{\phi(y,x')} * \delta_{h'}(H)]\delta_{y'}(Y) = 1$$

Since $\delta_h * \delta_{\phi(y,x')}$ is a probability measure with compact support in H, $\delta_h * \delta_{\phi(y,x')} * \delta_{h'}$ is a probability measure with compact support in H and it follows that $\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_h') \otimes \delta_{y'}$ is a probability measure with compact support in K. Next we have to show that $\bullet$ is associative. Let $(x,h,y),(x',h',y')$ and (x'',h'',y'') be three arbitrary elements of K then

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}] \bullet \delta_{(x'',h'',y'')} =$$

$$[\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}] \bullet \delta_{(x'',h'',y'')} =$$

$$\delta_x \otimes ((\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) * \delta_{\phi(y',x'')} * \delta_{h''}) \otimes \delta_{y''}$$

And

$$\delta_{(x,h,y)} \bullet [\delta_{(x',h',y')} \bullet \delta_{(x'',h'',y'')}] = \delta_{(x,h,y)} \bullet [\delta_{x'} \otimes \delta_{h'} * \delta_{\phi(y',x'')} * \delta_{h''} \otimes \delta_{y''} =$$

$$\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * (\delta_{h'} * \delta_{\phi(y',x'')} * \delta_{h''})) \otimes \delta_{y''}$$

now we can easily see that

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}] \bullet \delta_{(x'',h'',y'')} = \delta_{(x,h,y)} \bullet [\delta_{(x',h',y')} \bullet \delta_{(x'',h'',y'')}]$$

This shows that $(K,\bullet)$ is a semihypergroup.

Up to this point we have considered $\phi : Y \times X \longrightarrow H$ and have not used the fact that ϕ maps $Y \times X$ into Z, the center of H. We will require this in what follows.

3.2 Proposition

An element $(x,h,y) \in K$ is an idempotent element if and only if $h = \phi(y,x)^{-}$. Furthermore, idempotent elements of K are in its center.

Proof:

Let (x,h,y) be an idempotent element of K. Then, we have :

$$\delta_{(x,h,y)} \bullet \delta_{(x,h,y)} = \delta_x \otimes \delta_h * \delta_{\phi(y,x)} * \delta_h \otimes \delta_y = \delta_x \otimes \delta_h \otimes \delta_y$$

That is,

$$\delta_h * \delta_{\phi(y,x)} * \delta_h = \delta_h$$

Multiplying both sides of the equality above by $\delta_{\phi(y,x)}$ on the left, we have

$$(\delta_{\phi(y,x)} * \delta_h) * (\delta_{\phi(y,x)} * \delta_h) = \delta_{\phi(y,x)} * \delta_h$$

This shows that $(\delta_{\phi(y,x)} * \delta_h)$ is an idempotent element of the hypergroup H and so is the point mass at the identity of H, therefore, $h = \phi(y,x)^-$.

We note here that if we did not assume that $\phi(y,x)$ was in the center of H this result will still hold as $(\delta_{\phi(y,x)} * \delta_h)$ will be considered an idempotent probability measure and so its support is a subhypergroup (JE 10.2E) of H containing the identity so that $h = \phi(y,x)^-$, by axiom SH_6 in the definition of a hypergroup.

Next we need to show that $\forall x \in X$ and $y \in Y$, $(x, \phi(y,x)^-, y)$ is an idempotent element of K for

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x,\phi(y,x)^-,y)} =$$

$$\delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x)} * \delta_{\phi(y,x)^-} \otimes \delta_y =$$

$$\delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y = \delta_{(x,\phi(y,x)^-,y)}$$

since $\delta_{\phi(y,x)^-}$ is in the center of H(this is the first time we have used the center property of Z), let $(x, \phi(y,x)^-, y)$ and (x', h', y') be two arbitrary elements of K. Then,

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x',h',y')} =$$

$$\delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{h'} \otimes \delta_{y'}$$

Notice that by the center property of Z, $\delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{h'}$ is a point mass. Thus, $(x, \phi(y,x)^-, y)$ is in the center of K.

3.3 Theorem

If H is a hypergroup, and X and Y are two nonempty locally compact Hausdorff spaces, then the semihypergroup $K = X \times H \times Y$ with the Rees convolution product, as defined above, is completely simple.

Proof:

First, we need to show that K is simple. Let I be an ideal of K and let $(x,h,y) \in K$ be a point in K. We will show that $(x,h,y) \in I$ which shows that $K = I$. To do this, let (x_1,h_1,y_1) be any point of I. Then the support of the probability measure $\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)}$ is a subset of I. We will prove that the point $(x,h,y) \in I$.

By definition of the convolution product on K

$$\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)} =$$

$$\delta_x \otimes \delta_h * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h) \otimes \delta_y$$

And observe that

$$Supp(\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)}) = \{x\} \times Supp(\delta_h * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h)) \times \{y\}$$

Thus whenever $(x, h, y) \in K$,$\{x\} \times Supp(\delta_h * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h) \times \{y\} \subset I$ Since $\delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)}) = \delta_k$ for some $k \in H$,we have $(x, k^-, y) \in \{x\} \times Supp(\delta_{k^-} * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)}) * \delta_{k^-}) \times \{y\} \subset I$ for some $k \in H$. Now if $\delta_{k^-} * \delta_{\phi(y,x)} = \delta_u$, then $(x, u^-, y) \in K$ and $(x, e, y) \in \{(x, k^-, y)\} \bullet \{(x, u^-, y)\} \subset I$. Now for any $h \in H$, $(x, \{\phi(y, x)^-\} * \{h\}, y) \in K$ and we have

$$(x, e, y) \bullet (x, \{\phi(y, x)^-\} * \{h\}, y) = (x, h, y) \in I.$$

This shows that $I = K$, and thus K is simple.

Next we need to show that K contains a primitive idempotent element. Now suppose $(x, \phi(y, x)^-, y)$ and $(x', \phi(y', x')^-, y')$ are two idempotent elements of K such that $(x, \phi(y, x)^-, y) \leq (x', \phi(y', x')^-, y')$ then

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x',\phi(y',x')^-,y')} = \delta_{(x,\phi(y,x)^-,y)}$$

which is equivalent to

$$\delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{\phi(y',x')^-} \otimes \delta'_y = \delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y$$

so that $y' = y$

And

$$\delta_{(x',\phi(y',x')^-,y')} \bullet \delta_{(x,\phi(y,x)^-,y)} = \delta_{(x,\phi(y,x)^-,y)}$$

which is equivalent to

$$\delta_{x'} \otimes \delta_{\phi(y',x')^-} * \delta_{\phi(y',x)} * \delta_{\phi(y,x)^-} \otimes \delta_y = \delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y$$

so that $x' = x$. Combining these two results we see that $(x, \phi(y, x)^-, y) = (x', \phi(y', x')^-, y')$ That is $(x', \phi(y', x')^-, y')$ is a minimal idempotent element. And similarly we can show that $(x, \phi(y, x)^-, y)$ is a minimal idempotent element. So all idempotent elements of K are primitives, so K is a completely simple semihypergroup.

3.4 Theorem

Let $(H,*)$ be a hypergroup and s,t two elements. Then $\{s\} \times H \times \{t\}$ with the Rees convolution product is a cell hypergroup with identity element $(s,\phi(t,s)^-,t)$ and the involution defined by $(s,h,t)^\vee = (s,h',t)$ if and only if

$$\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$$

Proof:

First we need to show that $(s,\phi(t,s)^-,t)$ is the identity of $\{s\} \times H \times \{t\}$ Let $h \in H$ then

$$\delta_{(s,\phi(t,s)^-,t)} \bullet \delta_{(s,h,t)} =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_h \otimes \delta_t =$$

$$\delta_s \otimes \delta_h \otimes \delta_t$$

And since $\delta_{\phi(t,s)^-} * \delta_{\phi(t,s)}$ is the identity in H this equality holds.

Next we need to show that for all $h \in H$, $(s,h,t)^{\vee\vee} = (s,h,t)$, and $(s,\phi(t,s)^-,t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)})$ if and only if $(s,h,t)^\vee = (s,h',t)$.

Suppose $(s,h,t)^\vee = (s,h',t)$ where

$$\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$$

Suppose also that $(s,h',t)^\vee = (s,h'',t)$ where

$$\delta_{h''} = \delta_{\phi(t,s)^-} * \delta_{h'^-} * \delta_{\phi(t,s)^-}$$

then

$$\delta_{h'^-} = \delta_{\phi(t,s)} * \delta_h * \delta_{\phi(t,s)}$$

So that

$$\delta_{h''} = \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_h * \delta_{\phi(t,s)} * \delta_{\phi(t,s)^-} = \delta_h$$

So $h = h''$ and therefore $(s,h,t)^{\vee\vee} = (s,h,t)$

Next suppose $(s,\phi(t,s)^-,t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)}$, that is $\phi(t,s)^- \in \{h\} * \{\phi(t,s)\} * \{h'\}$ which is equivalent to $h' \in \{\phi(t,s)^-\} * \{h^-\} * \{\phi(t,s)^-\}$ but $\{\phi(t,s)^-\} * \{h^-\} * \{\phi(t,s)^-\}$ is a singleton as $\phi(t,s)^-$ is in the center of H so $\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$ which shows that $(s,h,t)^\vee = (s,h',t)$.

Now suppose $(s,h,t)^\vee = (s,h',t)$ then $\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$ which implies that $h' \in \phi(t,s)^- * \{h^-\} * \{\phi(t,s)^-\}$ which is equivalent to $\phi(t,s)^- \in \{h\} * \{\phi(t,s)\} * \{h'\}$ which shows that $(s,\phi(t,s)^-,t) \in \{s\} \times \{h\} * \{\phi(t,s)\} * \{h'\} \times \{t\}$ That is $(s,\phi(t,s)^-,t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)})$.

Next we need to show that

$$(\delta_{(s,h,t)} \bullet \delta_{(s,g,t)})^{\vee} = \delta_{(s,g,t)^{\vee}} \bullet \delta_{(s,h,t)^{\vee}}$$

Note that by the definition of involution on the $\{s\} \times H \times \{t\}$, if $\mu \in M(H)$ then

$$(\delta_s \otimes \mu \otimes \delta_t)^{\vee} = \delta_s \otimes \delta_{\phi(t,s)^-} * \mu^- * \delta_{\phi(t,s)^-} \otimes \delta_t$$

Now

$$(\delta_{(s,h,t)} \bullet \delta_{(s,g,t)})^{\vee} = (\delta_s \otimes \delta_h * \delta_{\phi(t,s)} * \delta_g \otimes \delta_t)^{\vee} =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * (\delta_h * \delta_{\phi(t,s)} * \delta_g)^{-} \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_g^- * \delta_{\phi(t,s)^-} * \delta_h^- * \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_g^- * \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_{\phi(t,s)^-} * \delta_h^- * \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{g'} * \delta_{\phi(t,s)} * \delta_{h'} \otimes \delta_t =$$

$$\delta_{(s,g,t)^{\vee}} \bullet \delta_{(s,h,t)^{\vee}}$$

Which completes the proof.

4 Other sets products for topological semihypergroups

4.1 Definition

Let S be a locally compact semihypergroup and B be a Borel subset of S. Then

$$Bx^- = \{y \in S : Supp(\delta_y * \delta_x) \cap B \neq \emptyset\}$$

Similarly,

$$x^- B = \{y \in S : Supp(\delta_x * \delta_y) \cap B \neq \emptyset\}$$

4.2 Theorem

Let B be a Borel subset of a locally compact second countable semihypergroup S. Then for any $x \in S$, the sets Bx^- and x^-B are also Borel subsets of S.

Proof We only prove that Bx^- is Borel whenever B is Borel, the other side follows in a similar way. To this end, first notice that if B is open, then we have:

$$Supp(\delta_y * \delta_x) \cap B \neq \emptyset$$

implies that $\delta_y * \delta_x(B) > 0$. Since the map $(x, y) \longmapsto \delta_y * \delta_x$ is a continuous map (with respect to weak topology in $M_1(S)$) by axiom SH_3, there is an open subset $N(y)$ containing y such that for each $y' \in N(y)$, $\delta_{y'} * \delta_x(B) > 0$. This means that

$$Supp(\delta_{y'} * \delta_x) \cap B \neq \emptyset$$

for each $y' \in N(y)$ so that $N(y) \subset Bx^-$; consequently, Bx^- is open whenever B is open. Let us now suppose that B is a closed subset of S. Let $x \in S$ and $y \in (Bx^-)^c$. Then we have:

$$Supp(\delta_y * \delta_x) \cap B = \emptyset$$

so that $Supp(\delta_y * \delta_x)$, which is compact, is contained in the open set B^c. Since by SH_4, the map $(x, y) \longmapsto \delta_y * \delta_x$ is continuous with respect to the product topology in the domain and the Michael topology for the compact subsets in the range, the set

$$\{y' : Supp(\delta_{y'} * \delta_x) \subset B^c\}$$

is an open set containing y; in other words, $(Bx^-)^c$ is open, and this means that Bx^- is closed whenever B is closed.

Now let us define the class $\mathcal{F}$ by

$\mathcal{F} = \{B : Bx^-$ is Borel whenever B is Borel and $x \in S\}$. Then $\mathcal{F}$ contains all open and all closed subsets of S. Furthermore, if V is an open set and W is a closed set, then since S is locally compact Hausdorff second countable, there is a sequence $\{F_n\}$ of closed sets such that $V = \bigcup_{n=1}^{\infty} F_n$.

$$[(V \cap W)x^-]^c = \{y : Supp(\delta_y * \delta_x) \cap (V \cap W) = \emptyset\} =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup \{y : Supp(\delta_y * \delta_x) \cap W \subset V^c\} =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup [\bigcap_{n=1}^{\infty} \{y : Supp(\delta_y * \delta_x) \cap W \subset F_n^c\}] =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup [\bigcap_{n=1}^{\infty} \{\{y : Supp(\delta_y * \delta_x) \subset F_n^c \cup W^c\} - \{y : Supp(\delta_y * \delta_x) \subset W^c\}\}]$$

Now the mapping $\gamma : S \times S \longrightarrow \mathcal{C}(S) : (y, x) \longmapsto Supp(\delta_y * \delta_x)$ is continuous, and since the sets $\mathcal{C}_S(W^c)$, $\mathcal{C}_S(F_n^c \cup W^c)$ are open sets (in the Michael topology, $\gamma^{-1}(\mathcal{C}_S(W^c)) = \{y : Supp(\delta_y * \delta_x) \subset W^c\}$ and $\gamma^{-1}(\mathcal{C}_S(F_n^c \cup W^c)) = \{y : Supp(\delta_y * \delta_x) \subset F_n^c \cup W^c\}$ are open, so are Borel sets. It follows that $[(V \cap W)x^-]^c$ is a Borel set. Therefore, $(V \cap W)x^-$ is a Borel set.

This means that the algebra $\mathcal{A}$ (finite intersections and complements) generated by all open subsets of S is contained in $\mathcal{F}$. It is also clear that

$$(\bigcup_{n=1}^{\infty} B_n)x^- = \bigcup_{n=1}^{\infty} (B_n x^-)$$

whenever $B_n \in \mathcal{F}, n \geq 1$, and $x \in S$. This means that the monotone class generated by $\mathcal{A}$, which is a σ-algebra and which contains all Borel subsets of S, is contained in $\mathcal{F}$. Which completes the proof.

4.3 Proposition

Let S be a locally compact semihypergroup, $B \subset S$ and $x \in S$. Then

$$B \subset (Bx)x^-$$

Proof

$(Bx)x^- = \{y \in S : Supp(\delta_y * \delta_x) \cap Bx \neq \emptyset\}$ Since $Bx = \bigcup_{b \in B} Supp(\delta_b * \delta_x)$, If $y \in B$, then $Supp(\delta_y * \delta_x) \subset Bx$; therefore, $Supp(\delta_y * \delta_x) \cap Bx \neq \emptyset$ so $y \in (Bx)x^-$, which implies $B \subset (Bx)x^-$

4.4 Proposition

Let S be a locally compact semihypergroup and C be a compact subset of S. If $B \subset S$,

$$(B - Cx)x^- \subset Bx^- - C$$

Proof:

If $y \in (B-Cx)x^-$ then $Supp(\delta_y * \delta_x) \cap (B-Cx) \neq \emptyset \Longrightarrow Supp(\delta_y * \delta_x) \cap B \neq \emptyset$ and $Supp(\delta_y * \delta_x) \cap (Cx)^c \neq \emptyset \Longrightarrow y \in Bx^-$ and $Supp(\delta_y * \delta_x)$ is not entirely in Cx that is $y \notin C$ (for if $y \in C$ then $Supp(\delta_y * \delta_x) \subset Cx) \Longrightarrow y \in Bx^- - C \Longrightarrow (B - Cx)x^- \subset (Bx^- - C)$.

The next result was proved for hypergroups in [BH95]. The same result holds for semihypergroups with the same proof which we reproduce here.

4.5 Proposition

Let S be a locally compact space and $\mu \in M_1(S)$. Then $\forall x \in S$ and compact $C \subset S$

$$\delta_x * \mu(C) \leq \mu(x^- C)$$

Proof:

By definition,

$$x^- C = \{y \in S : Supp(\delta_x * \delta_y) \cap C \neq \emptyset\}$$

So $y \in x^- C$ if and only if $Supp(\delta_x * \delta_y) \cap C \neq \emptyset$. Thus,

$$\delta_x * \mu(C) = \int_S \delta_x * \delta_y(C)\mu(dy) = \int_{x^- C} \delta_x * \delta_y(C)\mu(dy) \leq \mu(x^- C)$$

since $\delta_x * \delta_y(C) \leq 1$

Remark

As pointed out in [BH95] we cannot expect equality here even when S is compact.

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