International Mathematical Forum, Vol. 7, 2012, no. 36, 1751 - 1760

Inclusion Properties of Certain Subclasses of Analytic Functions

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Abstract

The purpose of the present paper is to introduce new subclasses of analytic functions and to investigate inclusion properties of these classes, using the principle of subordination. Also inclusion properties of these classes involving the generalized integral operator are obtained.

Mathematical Subject Classification: 30C45

Keywords: Analytic function, multiplier transformation, differential subordination

1. INTRODUCTION

Let A denote the class of functions of the form

(1.1) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that

$f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence(See [4],[10] and [11]):

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $0 \le \xi, \rho < 1$, we denote by $S^*(\xi), K(\xi)$ and $C(\xi, \rho)$ the subclasses of A consisting of all analytic functions which are, respectively, starlike of order ξ, convex of order ξ, and close-to-convex of order ρ and type ξ in U [10, 14].

Let Λ be the class of all functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\operatorname{Re}(\phi(z)) > 0, z \in U$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S^*(\xi;\phi), K(\xi;\phi)$ and $C(\xi, \rho; \phi, \varphi)$ of the class A, $0 \le \xi < 1$, $0 \le \rho < 1$ and $\phi, \varphi \in \Lambda$, which are defined by:

$$S^*(\xi;\phi) = \left\{ f \in A : \frac{1}{1-\xi}\left(\frac{zf'(z)}{f(z)} - \xi \right) \prec \phi(z), z \in U \right\},$$

$$K(\xi;\phi) = \left\{ f \in A : \frac{1}{1-\xi}\left(1 + \frac{zf''(z)}{f'(z)} - \xi \right) \prec \phi(z), z \in U \right\}, \text{ and}$$

$$C(\xi, \rho; \phi, \varphi) = \left\{ f \in A : \frac{1}{1-\rho}\left(\frac{zf'(z)}{g(z)} - \rho \right) \prec \varphi(z), z \in U, where, g \in S^*(\xi;\phi) \right\}.$$

From these definitions, we can obtain some well-known subclasses of A, by special choices of the functions ϕ and φ. For example, we have

$$S^*\left(\xi; \frac{1+z}{1-z} \right) = S^*(\xi),\ K\left(\xi; \frac{1+z}{1-z} \right) = K(\xi) \text{ and } C\left(\xi, \rho; \frac{1+z}{1-z}, \frac{1+z}{1-z} \right) = C(\xi, \rho).$$

Now, we define a new generalized operator $I^n_{\alpha,\beta}$ on A as below:

Definition 1.1. Let n$\in N_0 = N \cup \{0\}$, $\beta \ge 0$ and α a real number with $\alpha + \beta > 0$. Then for $f \in A$, we define the operator $I^n_{\alpha,\beta}$ by

$$I^0_{\alpha,\beta} f(z) = f(z),$$

$$I^1_{\alpha,\beta} f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta},$$

$$\ldots,$$

$$I^n_{\alpha,\beta} f(z) = I_{\alpha,\beta}(I^{n-1}_{\alpha,\beta} f(z)).$$

Remark 1.2. We observe that $I^{n}_{\alpha,\beta}: A \rightarrow A$, is a linear operator and for $f(z)$ given by (1.1), we have

(1.2) $$I^{n}_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^{n} a_k z^k .$$

It follows from (1.2) that

(1.3) $$I^{n}_{\alpha,0} f(z) = f(z),$$

(1.4) $$(\alpha + \beta) I^{n+1}_{\alpha,\beta} f(z) = \alpha I^{n}_{\alpha,\beta} f(z) + \beta z (I^{n}_{\alpha,\beta} f(z))^{'}, \beta > 0,$$

and

$$I^{n_1}_{\alpha,\beta} (I^{n_2}_{\alpha,\beta} f(z)) = I^{n_2}_{\alpha,\beta} (I^{n_1}_{\alpha,\beta} f(z)), \text{ for all } n_1, n_2 \in N_0 .$$

We note that

- $I^{n}_{\alpha,1} f(z) = I^{n}_{\alpha} f(z), \alpha > -1$ (See Cho and Srivastava [6] and Cho and Kim [7]).
- $I^{n}_{1-\beta,\beta} f(z) = D^{n}_{\beta} f(z), \beta \geq 0$ (See Al-Oboudi [1]).
- $I^{n}_{l+1-\beta,\beta} f(z) = I^{n}_{l,\beta} f(z), l > -1, \beta \geq 0$ (See Catas [5]).
- $I^{n}_{1,\beta} f(z) = N^{n}_{\beta} f(z)$, is a new operator defined by

$$N^{n}_{\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + k\beta}{1 + \beta} \right)^{n} a_k z^k \ \ (f \in A, \beta \geq 0)$$

Remark 1.3. a) i) $I^{n}_{\alpha} f(z)$ was investigated in [6] and [7] for $\alpha \geq 0$ and $I^{n}_{l,\beta} f(z)$ was defined in [5] for $l \geq 0, \beta \geq 0$. So our results in this paper are improvement of corresponding results proved earlier for $I^{n}_{\alpha} f(z)$ or $I^{n}_{l,\beta} f(z)$ to $\alpha > -1$ or $l. > -1$, respectively.

b) i) $D^{n}_{1} f(z)$ was introduced by Salagean [13] and was considered for $n \geq 0$ in [3], and ii) $I^{n}_{1} f(z)$ was due to Uralegaddi and Somanath [15].

Next, by using the operator $I^{n}_{\alpha,\beta}$, we introduce the following subclasses of analytic functions for $\phi, \varphi \in \Lambda, n \in N_0, \beta \geq 0, \alpha$ a real number with $\alpha + \beta > 0$, $0 \leq \xi < 1$ and $0 \leq \rho < 1$:

$$S^{n}_{\alpha,\beta}(\xi;\phi) = \left\{ f \in A : I^{n}_{\alpha,\beta} f(z) \in S^{*}(\xi;\phi) \right\},$$

$$K^{n}_{\alpha,\beta}(\xi;\phi) = \left\{ f \in A : I^{n}_{\alpha,\beta} f(z) \in K(\xi;\phi) \right\},$$

and

$$C^{n}_{\alpha,\beta}(\xi,\rho;\phi,\varphi) = \left\{ f \in A : I^{n}_{\alpha,\beta} f(z) \in C(\xi,\rho;\phi,\varphi) \right\}.$$

We also note that

(1.5) $$f(z) \in K^{n}_{\alpha,\beta}(\xi;\phi) \Leftrightarrow z f^{'}(z) \in S^{n}_{\alpha,\beta}(\xi;\phi) .$$

In particular, we set

(1.6) $$S^n_{\alpha,\beta}\left(\xi;\frac{1+\mathrm{A}z}{1+\mathrm{B}z}\right)=S^n_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B}),-1\le \mathrm{B}<\mathrm{A}\le 1,$$

and

(1.7) $$K^n_{\alpha,\beta}\left(\xi;\frac{1+\mathrm{A}z}{1+\mathrm{B}z}\right)=K^n_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B}),-1\le \mathrm{B}<\mathrm{A}\le 1.$$

In section 2, some preliminary results are mentioned. In section 3, we show that $S^{n+1}_{\alpha,\beta}(\xi;\phi)\subset S^n_{\alpha,\beta}(\xi;\phi)$, $K^{n+1}_{\alpha,\beta}(\xi;\phi)\subset K^n_{\alpha,\beta}(\xi;\phi)$ and $C^{n+1}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)\subset C^n_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$. In section 4, we study inclusion properties of classes $S^n_{\alpha,\beta}(\xi;\phi), K^n_{\alpha,\beta}(\xi;\phi)$ and $C^n_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$, involving generalized Libera integral operator.

2. PRELIMINARY LEMMAS

The following lemmas will be required in our investigation.

Lemma 2.1 ([8]). Let ϕ be convex, univalent in U with $\phi(0)=1$ and $\mathrm{Re}(\kappa\phi(z)+\gamma)>0$, $\kappa,\gamma\in C$. If p is analytic in U with $p(0)=1$, then

$$p(z)+\frac{zp'(z)}{\kappa p(z)+\gamma}\prec\phi(z),(z\in U)\text{ implies } p(z)\prec\phi(z),(z\in U).$$

Lemma 2.2 ([10]). Let ϕ be convex, univalent in U and w be analytic in U with $\mathrm{Re}(w(z))\ge 0$. If p is analytic in U with $p(0)=\phi(0)$, then

$$p(z)+w(z)zp'(z)\prec\phi(z),(z\in U)\text{ implies } p(z)\prec\phi(z),(z\in U).$$

3. INCLUSION PROPERTIES INVOLVING THE OPERATOR $I^n_{\alpha,\beta}$.

Unless otherwise mentioned we shall assume that $\beta\ge 0,\alpha$ a real number with $\alpha+\beta>0$, $n\in N_0, 0\le\xi<1$ and $0\le\rho<1$, throughout this paper.

Theorem 3.1. Let $f\in A$ and let $\phi\in\Lambda$ with $\mathrm{Re}((1-\xi)\phi(z)+\xi+(\alpha/\beta))>0$. Then
$$S^{n+1}_{\alpha,\beta}(\xi;\phi)\subset S^n_{\alpha,\beta}(\xi;\phi).$$

Proof. Let $f(z)\in S^{n+1}_{\alpha,\beta}(\xi;\phi)$ and set

(3.1) $$p(z)=\frac{1}{1-\xi}\left(\frac{z(I^{n}_{\alpha,\beta}f(z))'}{I^{N}_{\alpha,\beta}f(z)}-\xi\right),$$

where $p(z)$ is analytic in U with $p(0)=1$. Using (1.4) in (3.1), we get

(3.2) $$\left(\frac{\alpha+\beta}{\beta}\right)\frac{I^{n+1}_{\alpha,\beta}f(z)}{I^{n}_{\alpha,\beta}f(z)}=(1-\xi)p(z)+\xi+(\alpha/\beta).$$

Differentiating (3.2) logarithmically with respect to z, we obtain

(3.3) $$\frac{1}{1-\xi}\left(\frac{z(I^{n+1}_{\alpha,\beta}f(z))'}{I^{n+1}_{\alpha,\beta}f(z)}-\xi\right)=p(z)+\frac{zp'(z)}{(1-\xi)p(z)+\xi+(\alpha/\beta)},z\in U.$$

Applying Lemma 2.1 to (3.3), it follows that $p\prec\phi$, i.e $f\in S^{n}_{\alpha,\beta}(\xi;\phi)$.

Theorem 3.2. Let $f\in A$ and let $\phi\in\Lambda$ with $\mathrm{Re}((1-\xi)\phi(z)+\xi+(\alpha/\beta))>0$. Then

$$K^{n+1}_{\alpha,\beta}(\xi;\phi)\subset K^{n}_{\alpha,\beta}(\xi;\phi).$$

Proof. Applying (1.5) and Theorem 3.1, we conclude that

$$\begin{aligned} f\in K^{n+1}_{\alpha,\beta}(\xi;\phi) &\Rightarrow zf'\in S^{n+1}_{\alpha,\beta}(\xi;\phi)\subset S^{n}_{\alpha,\beta}(\xi;\phi)\\ &\Leftrightarrow zf'\in S^{n}_{\alpha,\beta}(\xi;\phi)\\ &\Rightarrow f\in K^{n}_{\alpha,\beta}(\xi;\phi). \end{aligned}$$

Taking $\phi(z)=\frac{1+\mathrm{A}z}{1+\mathrm{B}z},-1\le\mathrm{B}<\mathrm{A}\le1,z\in U$, in Theorem 3.1 and Theorem 3.2, we have from (1.6) and (1.7), the following corollary.

Corollary 3.3. For $f\in A$, we have $S^{n+1}_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})\subset S^{n}_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})$ and $K^{n+1}_{\alpha,\beta}(\xi,,\mathrm{A},\mathrm{B})\subset$ $K^{n}_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})$.

By using Lemma 2.2, we obtain the following inclusion relation for the class $C^{n}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$.

Theorem 3.4. Let $f\in A$ and let $\phi,\varphi\in\Lambda$ with $\mathrm{Re}((1-\xi)\phi(z)+\xi+(\alpha/\beta))>0$. Then

$$C^{n+1}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)\subset C^{n}_{\alpha,\beta}(\xi,\rho;\phi,\varphi).$$

Proof. Let $f\in C^{n+1}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$, then by definition there exists a function $g\in S^{n+1}_{\alpha,\beta}(\xi;\phi)$ such that

$$\frac{1}{1-\rho}\left(\frac{z(I_{\alpha,\beta}^{n+1}f(z))'}{I_{\alpha,\beta}^{n+1}g(z)}-\rho\right)\prec\varphi(z), z\in U.$$

Now, let

$$p(z)=\frac{1}{1-\rho}\left(\frac{z(I_{\alpha,\beta}^{n+1}f(z))'}{I_{\alpha,\beta}^{n}g(z)}-\rho\right),$$

where $p(z)$ is analytic in U with $p(0)=1$.Using (1.4), we have

(3.4) $$\left(\frac{\alpha+\beta}{\beta}\right)I_{\alpha,\beta}^{n+1}f(z)=(\alpha/\beta)I_{\alpha,\beta}^{n}f(z)+[(1-\rho)p(z)+\rho]I_{\alpha,\beta}^{n}g(z).$$

Differentiating (3.4) with respect to z and multiplying by z, we get

(3.5) $$\left(\frac{\alpha+\beta}{\beta}\right)z(I_{\alpha,\beta}^{n+1}f(z))'=(\alpha/\beta)z(I_{\alpha,\beta}^{n}f(z))'+[(1-\rho)p(z)+\rho]z(I_{\alpha,\beta}^{n}g(z))'+$$
$$+(1-\rho)zp'(z)(I_{\alpha,\beta}^{n}g(z).$$

Since $g\in S_{\alpha,\beta}^{n+1}(\xi;\phi)$, then by Theorem 3.1, we have $g\in S_{\alpha,\beta}^{n}(\xi;\phi)$. Let

$$h(z)=\frac{1}{1-\xi}\left(\frac{z(I_{\alpha,\beta}^{n}g(z))'}{I_{\alpha,\beta}^{n}g(z)}-\xi\right).$$

Applying (1.4) again, we get

(3.6) $$\left(\frac{\alpha+\beta}{\beta}\right)\frac{I_{\alpha,\beta}^{n+1}g(z)}{I_{\alpha,\beta}^{n}g(z)}=(1-\xi)h(z)+\xi+(\alpha/\beta).$$

From (3.5) and (3.6), we have

$$\frac{1}{1-\rho}\left(\frac{z(I_{\alpha,\beta}^{n+1}f(z))'}{I_{\alpha,\beta}^{n+1}g(z)}-\rho\right)=p(z)+\frac{zp'(z)}{(1-\xi)h(z)+\xi+(\alpha/\beta)}, z\in U.$$

Since $0\le\xi<1$ and $h(z)\prec\phi(z)$ in U, then $\mathrm{Re}((1-\xi)h(z)+\xi+(\alpha/\beta))>0$. So by taking $w(z)=1/[(1-\xi)h(z)+\xi+(\alpha/\beta)]$ and applying Lemma 2.2, we can show that $p\prec\varphi$, so that $f\in C_{\alpha,\beta}^{n}(\xi,\rho;\phi,\varphi)$, which proves Theorem 3.4.

4. INCLUSION PROPERTIES INVOLVING THE INTEGRAL OPERATOR F_c

In this section we consider the generalized Libera integral operator F_c (See [2],[9] and[12]) defined by

(4.1) $$F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, (c > -1; f \in A),$$

Theorem 4.1. Let $c > -1$ and let $\phi \in \Lambda$ with $\mathrm{Re}((1-\xi)\phi(z) + \xi + c) > 0$. If $f \in S^n_{\alpha,,\beta}(\xi;\phi)$, then $F_c(f) \in S^n_{\alpha,\beta}(\xi;\phi)$.

Proof. Let $f \in S^n_{\alpha,\beta}(\xi;\varphi)$ and set

(4.2) $$p(z) = \frac{1}{1-\xi}\left(\frac{z(I^n_{\alpha,\beta}F_c(f)(z))'}{I^n_{\alpha,\beta}F_c(f)(z)} - \xi\right),$$

where p is analytic in U with $p(0) = 1$. From (4.1), we have

(4.3) $$z(I^n_{\alpha,\beta}F_c(f)(z))' = (c+1)I^n_{\alpha,\beta}f(z) - cI^n_{\alpha,\beta}F_c(f)(z).$$

Using (4.3) in (4.2), we get

(4.4) $$(c+1)\frac{I^n_{\alpha,\beta}f(z)}{I^n_{\alpha,\beta}F_c(f)(z)} = (1-\xi)p(z) + \xi + c.$$

Differentiating (4.4) logarithmically with respect to z, we obtain

(4.5) $$\frac{1}{1-\xi}\left(\frac{z(I^n_{\alpha,\beta}f(z))'}{I^n_{\alpha,\beta}f(z)} - \xi\right) = p(z) + \frac{zp'(z)}{(1-\xi)p(z) + \xi + c}.$$

Applying Lemma 2.1 to (4.5), we conclude that $F_c(f)(z) \in S^n_{\alpha,\beta}(\xi;\phi)$.

Similarly applying (1.5) and Theorem 4.1, we have the following result.

Theorem 4.2. Let $c > -1$ and let $\phi \in \Lambda$ with $\mathrm{Re}((1-\xi)\phi(z) + \xi + c) > 0$. If $f \in K^n_{\alpha,,\beta}(\xi;\phi)$, then $F_c(f) \in K^n_{\alpha,\beta}(\xi;\phi)$.

From Theorem 4.1 and Theorem 4.2, we have the following corollary.

Corollary 4.3. Let $f \in A$. If $f \in S^n_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})$ (or $K^n_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})$), then $F_c(f) \in S^n_{\alpha,\beta}(\xi,\mathrm{A},\mathrm{B})$ (or $K^n_{\alpha,\beta}(\xi;\mathrm{A},\mathrm{B})$).

Theorem 4.4. Let $c > -1$ and let $\phi, \varphi \in \Lambda$ with $\mathrm{Re}((1-\xi)\phi(z)+\xi+c) > 0$. If $f \in C^{n}_{\alpha,,\beta}(\xi,\rho;\phi,\varphi)$, then $F_c(f) \in C^{n}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$.

Proof. Let $f \in C^{n}_{\alpha,\beta}(\xi,\rho;\phi,\varphi)$. Then there exists a function $g \in S^{n}_{\alpha,\beta}(\xi;\phi)$ such that

$$\frac{1}{1-\rho}\left(\frac{z(I^{n}_{\alpha,\beta}f(z))'}{I^{n}_{\alpha,\beta}g(z)}-\rho\right) \prec \varphi(z), z \in U .$$

We set, $p(z) = \frac{1}{1-\rho}\left(\frac{z(I^{n}_{\alpha,\beta}F_c(f)(z))'}{I^{n}_{\alpha,\beta}F_c(g)(z)}-\rho\right)$, where p is analytic in U with $p(0)=1$.

Since $g \in S^{n}_{\alpha,\beta}(\xi;\phi)$, we have from Theorem 4.1, that $F_c(g) \in S^{n}_{\alpha,\beta}(\xi;\phi)$. Using (4.3) we obtain

$$[(1-\rho)p(z)+\rho]I^{n}_{\alpha,\beta}F_c(g)(z)+cI^{n}_{\alpha,\beta}F_c(f)(z) = (c+1)I^{n}_{\alpha,\beta}f(z) .$$

Then by simple calculations, we get

$$(c+1)\frac{z(I^{n}_{\alpha,\beta}f(z))'}{I^{n}_{\alpha,\beta}F_c(g)(z)} = [(1-\rho)p(z)+\rho][(1-\xi)h(z)+\xi+c]+(1-\rho)zp'(z) ,$$

where $h(z) = \frac{1}{1-\xi}\left(\frac{z(I^{n}_{\alpha,\beta}F_c(g)(z))'}{I^{n}_{\alpha,\beta}F_c(g)(z)}-\xi\right)$. Hence, we have

$$\frac{1}{1-\rho}\left(\frac{z(I^{n}_{\alpha,\beta}f(z))'}{I^{n}_{\alpha,\beta}g(z)}-\rho\right) = p(z)+\frac{zp'(z)}{(1-\xi)h(z)+\xi+c} .$$

The remaining part of the proof is similar to that of Theorem 3.4 and so we omit it.

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Received: January, 2012