On Combination Graphs

M. A. Seoud and M.N. Al-Harere

Department of Mathematics, Faculty of Science
Ain Shams University, Abbassia, Cairo, Egypt
m.a.seoud@hotmail.com
manal.najy@yahoo.com

Abstract. Here we give two theorems and represent some families of combination graphs.

Keywords: Combination graphs

0 Introduction

Hegde and shetty [2, 4] define a graph G with n vertices to be a permutation graph if there exists an injection f from the vertices of G to the set {1, 2, 3,...,n} such that the induced edge function g defined as $g(f(u), f(v)) = f(u)!/[f(u) - f(v)]!$, $f(u) > f(v)$ is injective. They say a graph G with n vertices is a combination graph if there exists an injection f from the vertices of G to the set {1,2,3,...,n} such that the induced edge function g defined as $g(f(u), f(v)) = f(u)!/[f(u) - f(v)]!$, $f(u) > f(v)$ is injective. They prove: $K_n$ is a permutation graph if and only if $n \leq 5$; $K_n$ is a combination graph if and only if $n \leq 2$; $C_n$ is a combination graph for $n > 3$, $K_{n,n}$ is a combination graph if and only if $n \leq 2$; $W_n$ is a not a combination graph for $n > 3$. They strongly believe that $W_n$ is a combination graph for $n > 6$ and all trees are combination graphs. Basker Babujee and Vishnupriya [1] prove that the following graphs are permutation graphs: $P_n$, $C_n$; stars; graphs obtained by adding a pendent edge to each edge of a star; graphs obtained by joining the center of two identical stars with an edge or a path of length 2; and complete binary trees with at least three vertices. Seoud and Anwar [5] give the number of edges in any maximal combination graph $G(n, q)$ if n is even or if n is odd, $n > 3$. They show that $K_{m,n}$ is a combination graph if and only if $n, m \leq 2$ or $m=1$, consequently the fan $F_m(n + 1,2n - 1)$ is not a combination graph for $n \leq 4$. They give a survey of all maximal combination graphs on n vertices and q edges such that $n \leq 6$. Also they give a necessary condition for a graph to be a permutation graph, a strong k...
-combination graph and a strong permutation graph and applied these results on the wheel \( W_n(n + 1, 2n) \) and the fan \( F_n(n + 1, 2n - 1) \).

Here, first we represent two theorems: (1) A graph \( G(n, q) \) having at least 6 vertices, such that 3 vertices are of degree 1, \( n - 1, n - 2 \) is not a combination graph. (2) A graph \( G(n, q) \) having at least 6 vertices, such that there exist 2 vertices of degree \( n - 3 \), two vertices of degree 1 and one vertex of degree \( n - 1 \) is not a combination graph. Second, we show that the following families are combination graphs: Two copies of \( C_n \) sharing a common edge, the graph consisting of two cycles of the same order joined by a path of \( l \) vertices, the union of three cycles of the same order, the wheel \( W_n \), \( n \geq 7 \), what Hegde and Shetty believed, the corona \( T_n \otimes K_1 \), where \( T_n \) is the triangular snake, the graph obtained from the gear \( G_{mr} \), by attaching \( n \) pendant vertices to each vertex which is not joined to the center of the gear, and some corollaries.

Any notion or definition which is not found here could be found in [3].

1. Two theorems

**Lemma 1.1.** In a combination graph the vertex of degree \( n - 1 \) receives label 1 or 2.

**Proof:** Let \( f \) be a combination labeling of \( (n, q) \) - graph. Then there exist vertices \( v_1, ..., v_n \) such that \( f(v_1) = 1, f(v_2) = 2, ..., f(v_n) = n \).

If we label the vertex of degree \( n - 1 \) by a number greater than 2 we will obtain two vertices \( v_i \) and \( v_j \) such that \( f(v_i) + f(v_j) = f(v_k) \), where \( v_k \) is the vertex of degree \( n - 1 \).

Hence we will get two edges with the same labels and this is a not combination graph.

**Remark 1.2**

1. The vertex \( v \) in the combination graph \( G(n, q) \) could be labeled by \( k \) if

\[
d(v) \leq \left\lfloor \frac{k}{2} \right\rfloor + n - k, k = 1, 2, ..., n.
\]

Hence if \( d(v) = n - r \), \( r = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor \), then \( n - r \leq \left\lfloor \frac{k}{2} \right\rfloor + n - k \) i.e. \( 2n - 2r \leq 2n - k \), hence \( k \leq 2r \).

2. The graph \( G(n, q) \) is not a combination graph if it has no vertex of degree \( \leq \left\lfloor \frac{n}{2} \right\rfloor \).

**Theorem 1.3.** A graph \( G(n, q) \) having at least 6 vertices, such that 3 vertices are of degree \( n - 1, n - 2, 1 \) is not a combination graph.
Proof. Let \( v, u, w \) be the vertices of degree \( n - 1, n - 2, 1 \) respectively. According to lemma 1.1 we have two cases:

Case 1: \( f(v) = 1, f(u) = k \),

Subcase (i) \( k = 2 \). It follows that \( w = 3 \)

Now we still have the vertices labeled 4, 6, and hence
\[
\binom{6}{1} = \binom{4}{2}.
\]

Subcase (ii): \( 2 < k < n \)

It follows that \( w = k - 1 \) or \( k + 1 \)

Hence we have either
\[
\binom{k+1}{k} = \binom{k+1}{1} \text{ or } \binom{k}{k-1} = \binom{k}{1}
\]

Subcase (iii) \( k = n \), here we have \( n - 2 > \left\lfloor \frac{n}{2} \right\rfloor \), \( n \geq 6 \).

Case 2: \( f(v) = 2, f(u) = k \)

Subcase (i) \( k = 1 \), hence \( f(w) = 3 \), but we have \( \binom{6}{1} = \binom{4}{2} \).

Subcase (ii) \( 2 < k < n \),

Hence \( f(w) = k - 2 \) or \( f(w) = k + 2 \), and we have either
\[
\binom{k+2}{k} = \binom{k+2}{2} \text{ or } \binom{k}{k-1} = \binom{k}{1}
\]
Sub case (iii) $k = n$, here we have $n - 2 > \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 6$.

In all cases the graph is not a combination graph.

**Remark**: For $n = 4$ or $n = 5$, the mentioned graph in the theorem 1.3 is a combination graph.

![Figure 1](image1)

*Figure (1)*

**Figure (2)**

**Theorem 1.3.** The graph $G(n, q)$ having 2 vertices of degree 1, 2 vertices of degree $n - 3$, 1 vertex of degree $n - 1$ is not a combination graph.

**Proof.** Let the vertices of degree $n - 3$ be labeled $k_1$ and $k_2$

Case 1: The vertex of degree $n - 1$ has the label 1

Subcase(i) $k_1 + k_2 \leq n$. Consider the labels $k_1 + 1, k_2 + 1, k_1 - 1, k_2 - 1, k_1 + k_2, k_1 - k_2$. Four of the vertices labeled with these labels would be joined to the vertices labeled by $k_1, k_2$ and all of them should be joined to the vertex labeled 1.

Subcase(ii) $k_1 + k_2 > n$. Similar to subcase (i), but the label $k_1 + k_2$ does not appear.

Case 2: The vertex of degree $n - 1$ has label 2.

Subcase(i)$k_1 + k_2 \leq n$. Consider the labels $k_1 + 2, k_2 + 2, k_1 - 2, k_2 - 2, k_1 + k_2, k_1 - k_2$. Four of the vertices labeled with these labels would be joined to the vertices labeled by $k_1, k_2$ and all of them should be joined to the vertex labeled 2.

Subcase(ii) $k_1 + k_2 > n$. Similar to subcase (i), but the label $k_1 + k_2$ does not appear.

The graph in all cases is not a combination graph.

**2 Some families**

**Theorem 2.1.** Two copies of $C_n$ sharing a common edge is a combination graph.

**Proof.** The graph is shown in Figure (3)

We define the labeling function

$f : V(G) \to \{1, 2, \ldots, 2n - 2\}$ as follows:
\[ f(v_i) = i, \quad i = 1, \ldots, 2n - 4, \]
\[ f(v_{2n-3}) = 2n - 2, \quad f(v_{2n-2}) = 2n - 3. \]

We need only to noted that \( \binom{2n-2}{2n-4} \neq \binom{2n-2}{n-2}. \)

**Definition and theorem 2.2.** The graph \( C_{m,l}^{++} \) consisting of two cycles of the same order \( m \), joined by a path of \( l \) vertices as shown in Figure (4) is a combination graph.

**Proof.** We define the labeling function
\[ f: V(C_{m,l}^{++}) \to \{ 1, 2, \ldots, 2m + l \} \]
as follows:
\[ f(v_i) = i, \quad i = 1, 2, \ldots, m - 1, \]
\[ f(v_m) = 2m + l, \quad f(w_j) = m - 1 + j, \quad j = 1, 2, \ldots, l, \]
\[ f(u_k) = m - 1 + l + k, \quad k = 1, 2, \ldots, m. \]

**Theorem 2.3.** The graph \( 3C_n \) is a combination graph.

**Proof.** Figure (5) shows the graph

![Figure (5)](image-url)
We define the function $f : V(3C_n) \rightarrow \{1, 2, ..., 3n\}$ as follows:

$f(v_i) = i, \quad i = 1, ..., n - 1, \quad f(v_n) = 3n - 1,$

$f(u_i) = n - 1 + i, \quad i = 1, ..., n - 1, \quad f(u_n) = 3n,$

$f(w_i) = 2n - 2 + i, \quad i = 1, ..., n, \quad f(w_n) = 3n - 2.$

It may be noted that: $\binom{3n-2}{2n-1} < \binom{3n-1}{n-1} < \binom{3n}{n} < \binom{3n}{3n-3}.$

**Definition and theorem 2.4.** The wheel $W_n$ is the graph $C_n + k_1, \quad n \geq 3.$ It is a combination graph for $n \geq 7.$

**Proof.**

We define the labeling function

$f : V(W_n) \rightarrow \{1, 2, ..., n + 1\}$ as follows:

**Case 1:** $n$ is odd, $n \geq 7$

$f(v_0) = 1,$

$f(v_i) = i + 1, \quad i = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor,$

$f(u_j) = \left\lfloor \frac{n}{2} \right\rfloor + 1 + j, \quad j = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor - 3,$

$f(u_{\left\lfloor \frac{n}{2} \right\rfloor - 2}) = n + 1, \quad f(u_{\left\lfloor \frac{n}{2} \right\rfloor - 1}) = n.$

(See Figure (6)).

Note that the labeling of the edges is increasing.

**Case 2:** $n$ is even, $n \geq 8$

$f(v_0) = 1, \quad f(v_i) = i + 1, \quad i = 1, 2, ..., \frac{n}{2}, \quad f(u_j) = \frac{n}{2} + 1 + j,$

$j = 1, ..., \frac{n}{2} - 3, \quad f\left( u_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) = n - 1, \quad f\left( u_{\frac{n}{2}}\right) = n + 1,$

It is easy to check that $\binom{\left\lfloor \frac{n}{2} \right\rfloor + 2}{2} \neq \binom{n+1}{2} \neq \binom{\left\lfloor \frac{n}{2} \right\rfloor + 3}{3}.$
Figure (7) shows a combination labeling of $W_{10}$

![Figure (7)](image)

**Corollary 2.5.** The graph $C_n^{(m)}$ (shown in Figure (8)) is a combination graph.

**Proof.** For $m=2$, $m=3$, $n=3$, it easy to fulfill the assertion. For $m \geq 4$ the graph $C_n^{(m)}$ is a subgraph of $W_n$ with the same vertices, since $W_n$ is a combination graph, $C_n^{(m)}$ is a combination graph.

![Figure (8)](image)

**Definition 2.6**

(a) Let $G_1$ and $G_2$ be two disjoint graphs. The corona $G_1 \odot G_2$ of $G_1$ and $G_2$ is the graph obtained by taking one copy of $G_1$ which has $n$ vertices and $n$ copies of $G_2$, and joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.

(b) The triangular snake $T_n$ is obtained from the path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to new vertices $v_i$ for $i = 1, \ldots, n-1$.

**Theorem 2.7.** The corona graph $T_n \odot K_1, n \geq 2$ is a combination graph.
Proof. Let $T_n \odot K_1$ be described as in Figure (9).

\[ \begin{array}{c}
\begin{array}{c}
\bullet v_1 \\
\downarrow \\
\bullet u_{n+1}
\end{array}
\begin{array}{c}
\downarrow \\
\bullet v_2 \\
\downarrow \\
\bullet u_{n+2}
\end{array}
\begin{array}{c}
\downarrow \\
\bullet v_3 \\
\downarrow \\
\bullet u_{n+3}
\end{array}
\begin{array}{c}
\cdots
\end{array}
\begin{array}{c}
\downarrow \\
\bullet u_2
\end{array}
\begin{array}{c}
\downarrow \\
\bullet u_1
\end{array}
\end{array}
\end{array} \]

Figure (9)

We define the labeling function

\[ f : V(T_n \odot K_1) \rightarrow \{ 1, \ldots, 4n - 2 \} \]

as follows:

\[ f ( v_i ) = 2i - 1 , \ i = 1, 2, \ldots, 2n - 1 \]

\[ f ( u_j ) = 2j , \ j = 1, 2, \ldots, 2n - 1 \]

We will divide the set of edge labels into three sets, which are:

\[ q_1 = \{ 2n, 2n + 1, 2n + 2, \ldots, 4n - 2 \} , \]

\[ q_2 = \left\{ \left( \binom{2n+1}{2}, \binom{2n+2}{2}, \binom{2n+4}{2}, \binom{2n+6}{2}, \ldots, \binom{4n-2}{2} \right) \right\} , \]

\[ q_3 = \left\{ \left( \binom{2n+2}{3}, \binom{2n+3}{3}, \binom{2n+4}{3}, \binom{2n+5}{3}, \ldots, \binom{4n-2}{2n-1} \right) \right\} . \]

The labeling in each set is increasing. Also we have:

\[ q_1 \cap q_2 = \emptyset , \text{ since } 4n - 2 < \binom{2n+1}{2} . \]

\[ q_1 \cap q_3 = \emptyset , \text{ since } 4n - 2 < \binom{2n+2}{3} . \]

\[ q_2 \cap q_3 = \emptyset , \text{ since } \binom{4n-2}{2} < \binom{2n+2}{3} . \ n \geq 2. \]

Example 2.8. Figure (10) shows a combination labeling of $T_6 \odot K_1$.
Definition and Theorem 2.9. We denote the graph obtained from the gear graph $G_m$ by attaching $n$ pendent vertices to each vertex which is joined to the center of the gear by $G_{n,m}$, as shown in Figure (11) is a combination graph.

Proof.
We define the labeling function

$f : V(G_{n,m}) \rightarrow \{1, 2, \ldots, 2m + nm + 1\}$ as follows:

$f(v_0) = 1$, $f(v_i) = i + 1$, $i = 1, 2, \ldots, m$,

$f(u_j) = m + j + 1$, $j = 1, 2, \ldots, m - 3$,

$f(u_{m-2}) = 2m$,

$f(u_{m-1}) = 2m - 1$,

$f(u_m) = 2m + 1$,

$f(w_j^{(i)}) = 2m + 1 + (j - 1)n + i$, $j = 1, 2, \ldots, n$, $i = 1, \ldots, m - 3$,

$f(w_j^{(m-2)}) = 2m + 1 + (m - 2)n + j$, $j = 1, 2, \ldots, n$,

$f(w_j^{(m-1)}) = 2m + 1 + (m - 3)n + j$, $j = 1, 2, \ldots, n$,

$f(w_j^{(m)}) = 2m + 1 + (m - 1)n + j$, $j = 1, 2, \ldots, n$.

Figure (12) shows a combination labeling of $G_{5,4}$.
Corollary 2.10.

(a) All gears are combination graphs.
(b) The graph resulting from $G_{n,m}$ after deleting the edges joining the center to the rim is a combination graph.

References


Received: January, 2012