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On Combination Graphs

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Abstract. Here we give two theorems and represent some families of combination graphs.

Keywords: Combination graphs

0 Introduction

Hegde and shetty [2, 4] define a graph G with n vertices to be a permutation graph if there exists an injection f from the vertices of G to the set $\{1, 2, 3,\ldots,n\}$ such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u) - f(v)|!$, $f(u) > f(v)$ is injective .They say a graph G with n vertices is a combination graph if there exists an injection f from the vertices of G to the set $\{1,2,3,\ldots,n\}$ such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$, $f(u) > f(v)$ is injective .They prove : K_n is a permutation graph if and only if $n \le 5$; K_n is a combination graph if and only if $n \le 2$; C_n is a combination graph for $n > 3$, $K_{n,n}$ is a combination graph if and only if $n \le 2$;W_n is a not a combination graph for $n \le 6$, and a necessary condition for a (p ,q)–graph to be a combination graph is that $4q \le p^2$ if p is even and $4q \le p^2 - 1$ if p is odd . They strongly believe that W_n is a combination graph for $n > 6$ and all trees are combination graphs. Basker Babujee and Vishnupriya [1] prove that the following graphs are permutation graphs : P_n ; C_n ; stars; graphs obtained by adding a pendent edge to each edge of a star ;graphs obtained by joining the center of two identical stars with an edge or a path of length 2 ; and complete binary trees with at least three vertices .Seoud and Anwar [5] give the number of edges in any maximal combination graph $G(n,q)$ if n is even or if n is odd , $n > 3$. They show that $K_{m,n}$ is a combination graph if and only if $n,m \le 2$ or m=1 , consequently the fan $F_n(n+1, 2n-1)$ is not a combination graph for $n \le 4$. They give a survey of all maximal combination graphs on n vertices and q edges such that $n \le 6$.Also they give a necessary condition for a graph to be a permutation graph , a strong k

-combination graph and a strong permutation graph and applied these results on the wheel $W_n(n+1,2n)$ and the fan $F_n(n+1,2n-1)$.

Here, first we represent two theorems;(1) A graph $G(n,q)$ having at least 6 vertices , such that 3 vertices are of degree $1, n-1, n-2$ is not a combination graph .(2) A graph $G(n,q)$ having at least 6 vertices , such that there exist 2 vertices of degree $n-3$, two vertices of degree 1 and one vertex of degree n-1 is not a combination graph . Second ,we show that the following families are combination graphs : Two copies of C_n sharing a common edge , the graph consisting of two cycles of the same order joined by a path of l vertices, the union of three cycles of the same order, the wheel W_n $n \geq 7$, what Hegde and shetty believed, the corona $T_n \odot K_1$, where T_n is the triangular snake , the graph obtained from the gear G_m , by attaching n pendent vertices to each vertex which is not joined to the center of the gear , and some corollaries .

Any notion or definition which is not found here could be found in [3].

1. Two theorems

Lemma1.1. In a combination graph the vertex of degree $n-1$ receives label 1 or 2.

Proof: Let f be a combination labeling of $(n, q)-$ graph. Then there exist vertices $v_1, \dots, v_n$ such that $f(v_1) = 1, f(v_2) = 2, \dots, f(v_n) = n$.

If we label the vertex of degree n-1 by a number greater than 2 we will obtain two vertices v_i and v_j such that $f(v_i) + f(v_j) = f(v_k)$, where v_k is the vertex of degree $n-1$.

Hence we will get two edges with the same labels and this is a not combination graph.

Remark 1.2

1. The vertex v in the combination graph G(n,q) could be labeled by k if

$d(v) \leq \left\lfloor \frac{k}{2} \right\rfloor + n - k$, $k = 1,2,\dots,n$. Hence if $d(v) = n - r$, $r = 1,2,\dots,\left\lfloor \frac{n}{2} \right\rfloor$, then $n - r \leq \left\lfloor \frac{k}{2} \right\rfloor + n - k$ i.e $2n - 2r \leq 2n - k$, *hence* $k \leq 2r$.

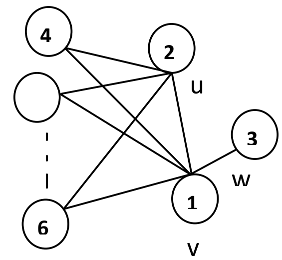
2. The graph G(n,q) is not a combination graph if it has no vertex of degree $\leq \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 1.3. A graph $G(n,q)$ having at least 6 vertices, such that 3 vertices are of degree $n-1, n-2, 1$ is not a combination graph.

Proof. Let $v\,,u\,,w$ be the vertices of degree $n-1, n-2\,,1$ respectively. According to lemma 1.1 we have two cases:

$Case$ 1: $f(v) = 1\,, f(\,u) = k$,

Subcase (i) $k = 2$. It follows that $w = 3$



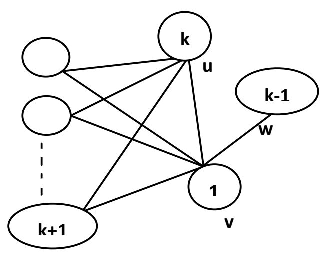
Now we still have the vertices labeled 4 , 6 , and hence

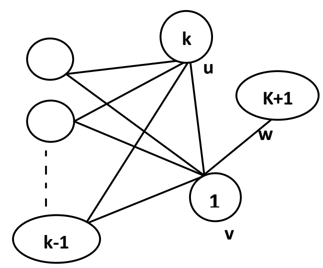
$\binom{6}{1} = \binom{4}{2}$.

Subcase (ii): $2 < \;k\; < n$

It follows that $w \;=\; k-1 \;\; or \;\; k+1$

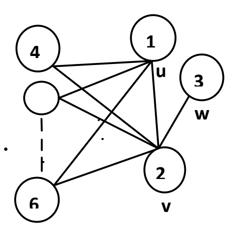
Hence we have either $\binom{k+1}{k} = \binom{k+1}{1} \; or \; \binom{k}{k-1} = \binom{k}{1}$





Subcase (iii) $k = n$, here we have $n-2 > \left\lfloor \frac{n}{2} \right\rfloor$, n≥ 6.

Case2 : $f(v) \;=\; 2\,, f\;(\;u\;) \;=\; k$

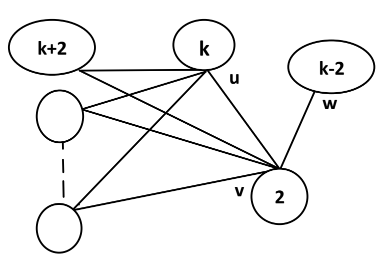


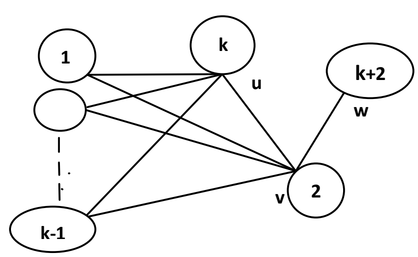
Subcase (i) $k = 1$, $hence\; f(\,w\,) = \;3$,but we have $\binom{6}{1} = \binom{4}{2}$.

Sub case (ii) $2 < k < n$,

Hence $f\;(\;w\;) \;=k-2$ or $f\;(\;w\;) \;=\; k+2$, and we have either

$\binom{k+2}{k} = \binom{k+2}{2} \;\; or \;\; \binom{k}{k-1} = \binom{k}{1}$





Sub case (iii) $k = n$, here we have $n - 2 > \left\lfloor \frac{n}{2} \right\rfloor$, n≥ 6.

In all cases the graph is not a combination graph.

Remark : For $n = 4 \; or \; n = 5$, the mentioned graph in the theorem 1.3 is a combination graph.

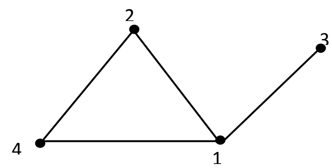


Figure (1)

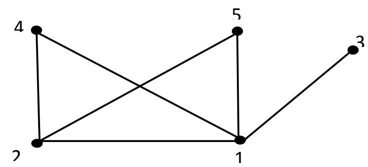


Figure (2)

Theorm1.3. The graph $G(n,q)$ having 2 vertices of degree 1, 2 vertices of degree $n-3$, 1 vertex of degree $n-1$ is not a combination graph.

Proof. Let the vertices of degree $n-3$ be labeled k_1 and k_2

Case 1: The vertex of degree $n-1$ has the label 1

Subcase(i) k_1+$k_2 \leq n$.Consider the labels $k_1+1, k_2+1, k_1-1, k_2-1, k_1+k_2, k_1-k_2$.Four of the vertices labeled with these labels would be joined to the vertices labeled by k_1 , k_2 and all of them should be joined to the vertex labeled 1.

Subcase(ii) k_1+ k_2>n. Similar to subcase (i), but the label k_1+k_2 does not appear.

Case 2: The vertex of degree $n-1$ has label 2.

Subcase(i)k_1+$k_2 \leq n$.Consider the labels $k_1+2, k_2+2, k_1-2, k_2-2, k_1+k_2, k_1-k_2$.Four of the vertices labeled with these labels would be joined to the vertices labeled by k_1, k_2 and all of them should be joined to the vertex labeled 2.

Subcase(ii) k_1+ k_2>n. Similar to subcase (i), but the label k_1+k_2 does not appear.

The graph in all cases is not a combination graph.

2 Some families

Theorem 2.1. Two copies of C_n sharing a common edge is a combination graph.

Proof. The graph is shown in Figure (3)

We define the labeling function

$f : V(G) \rightarrow \{1, 2, \ldots, 2n-2\}$ as follows:

$f(v_i) = i \quad , \quad i = 1, \dots, 2n-4,$

$f(v_{2n-3}) = 2n-2, \quad f(v_{2n-2}) = 2n-3.$

We need only to noted that $\binom{2n-2}{2n-4} \neq \binom{2n-2}{n-2}$.

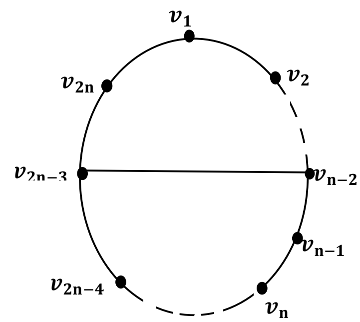


Figure (3)

Definition and theorem 2.2. The graph $C^{**}_{m,l}$ consisting of two cycles of the same order m, joined by a path of l vertices as shown in Figure (4) is a combination graph.

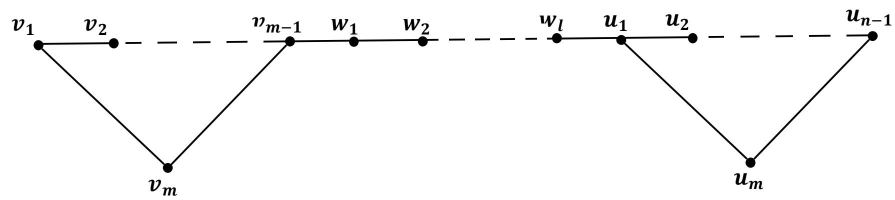


Figure (4)

Proof . we define the labeling function

$f: V(C^{**}_{m,l}) \rightarrow \{1, 2, \dots, 2m+l\}$ as follows: $f(v_i) = i, \quad i = 1,2,\dots, m-1,$
$f(v_m) = 2m+l \;,\; f(w_j) = m-1+j \quad , \quad j = 1,2,\dots, l,$

$f(u_k) = m-1+l+k \quad , \quad k = 1,2 \dots, m.$

Theorm2.3. The graph $3C_n$ is a combination graph.

Proof. Figure (5) shows the graph

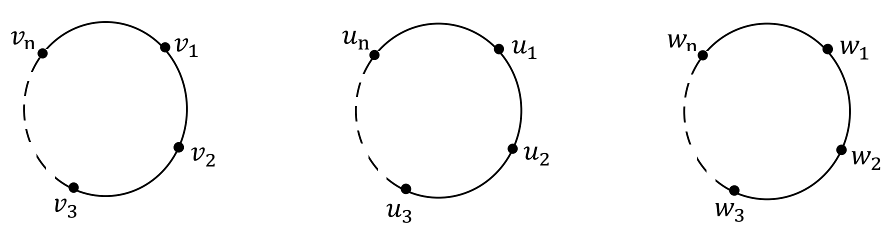


Figure (5)

We define the function $f : V(3C_n) \rightarrow \{1, 2, \dots, 3n\}$ as follows:

$f(v_i) = i \quad , \quad i = 1, \dots, n-1, f(v_n) = 3n - 1,$

$f(u_i) = n - 1 + i \quad , \quad i = 1, \dots, n-1 \ , \ f(u_n) = 3n,$

$f(w_i) = 2n - 2 + i \quad , \quad i = 1, \dots, n-, \ f(w_n) = 3n - 2.$

It may be noted that: $\binom{3n-2}{2n-1} < \binom{3n-1}{n-1} < \binom{3n}{n} < \binom{3n}{3n-3}$.

Definition and theorem 2.4. The wheel W_n is the graph $C_n + k_1$, $n \geq 3$. It is a combination graph for $n \geq 7$.

Proof.

We define the labeling function

$f : V(W_n) \rightarrow \{1, \dots, n+1\}$ as follows:

Case 1 : n is odd , $n \geq 7$

$f(v_0) = 1,$

$f(v_i) = i + 1 \quad , \quad i = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil \ ,$

$f(u_j) = \left\lceil \frac{n}{2} \right\rceil + 1 + j \quad , \ j = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil - 3 \ ,$

$f(u_{\lceil \frac{n}{2} \rceil - 2}) = n + 1 \ , \ f(u_{\lceil \frac{n}{2} \rceil - 1}) = n.$

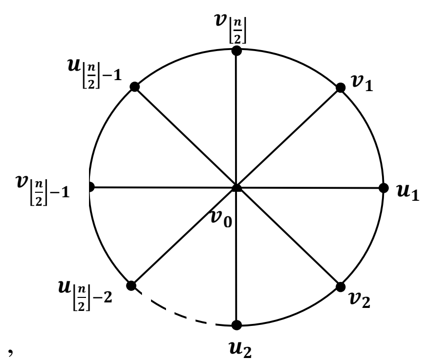


Figure (6)

(See Figure (6)).

Note that the labeling of the edges is increasing.

Case 2 : n is even , n≥ 8

$f(v_0) = 1, \ f(v_i) = i + 1 \ , \ i = 1, \dots, \frac{n}{2}, \quad f(u_j) = \frac{n}{2} + 1 + j \ ,$

$j = 1, \dots, \frac{n}{2} - 3, \ f\left(u_{(\frac{n}{2} - 1)}\right) = n - 1, \ f(u_{(\frac{n}{2})}) = n + 1,$

It is easy to check that $\binom{\frac{n}{2}+2}{3} \neq \binom{n+1}{2} \neq \binom{\frac{n}{2}+3}{3}$.

Figure (7) shows a combination labeling of W_{10}

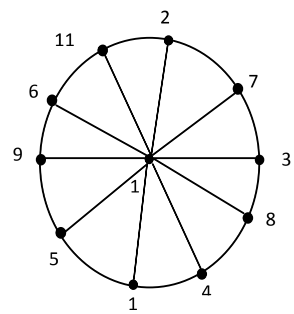


Figure (7)

Corollary 2.5. The graph $C_n^{(m)}$ (shown in Figure (8)) is a combination graph.

Proof . For m=2, m=3 , n=3, it easy to fulfill the assertion .For m$\geq$ 4 the graph $C_n^{(m)}$ is a subgraph of W_n with the same vertices , since W_n is a combination graph , $C_n^{(m)}$ is a combination graph .

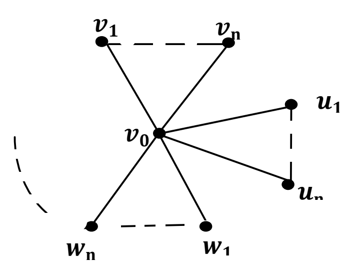


Figure (8)

Definition 2.6

(a)Let G_1 and G_2 be two disjoint graphs. The corona $G_1 \odot G_2$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 which has n vertices and n copies of G_2 , and joining the ith vertex of G_1 to every vertex in the ith copy of G_2 .

(b)The triangular snake T_n is obtained from the path $u_1 , u_2 , \ldots , u_n$ by joining u_i and u_{i+1} to new vertices v_i for $i = 1, \ldots, n-1$

Theorem 2.7. The corona graph $T_n \odot K_{1}, n \geq 2$ is a combination graph.

Proof . Let $T_n \odot K_1$ be described as in Figure (9).

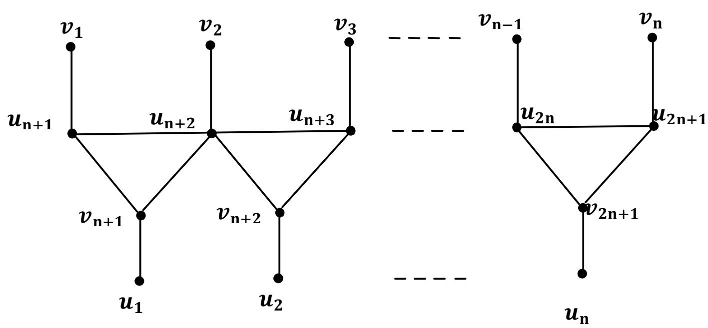


Figure (9)

We define the labeling function

f : V($T_n \odot K_1$) → { 1 , …,$4n-2$} as follows :

$f\ (v_i\ ) = \ 2i - 1\ ,\ \ i = 1,2, \ldots ,2n - 1\ ,\ \ f\ \left(u_j\right) = \ 2j\ ,\ \ j = 1,2, \ldots ,2n - 1.$

We will divide the set of edge labels into three sets, which are:

$q_1\ = \{\ 2n\ , 2n + 1, 2n + 2\ , \ldots ,4n - 2\ \},$

$q_2\ =\ \left\{\binom{2n+1}{2}, \binom{2n+2}{2}, \binom{2n+4}{2}, \binom{2n+6}{2}, \ldots, \binom{4n-2}{2}\right\},$

$q_3\ =\ \left\{\binom{2n+2}{3}, \binom{2n+3}{4}, \binom{2n+4}{5}, \ldots, \binom{4n-2}{2n-1}\right\}.$

The labeling in each set is increasing. Also we have:

$q_1 \cap q_2 = \emptyset\ , since\ 4n - 2 < \binom{2n+1}{2}\ ,q_1 \cap q_3 = \emptyset\ , since\ 4n - 2 < \binom{2n+2}{3}$

$q_2 \cap q_3 = \emptyset\ , since\ \binom{4n-2}{2} < \binom{2n+2}{3}\ ,n \geq 2.$

Example 2.8. Figure (10) shows a combination labeling of $T_6 \odot K_1$

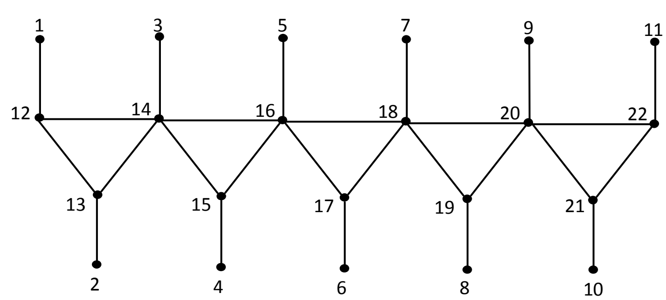


Figure (10)

Definition and Theorem 2.9. We denote the graph obtained from the gear graph G_m by attaching n pendent vertices to each vertex which is joined to the center of the gear by $G_{n,m}$, as shown in Figure (11) is a combination graph.

Proof.

We define the labeling function

$f : V(G_{n,m}) \to \{1,2, \dots, 2m+nm+1\}$ as follows:

$f(v_0) = 1, f(v_i) = i+1, \; i = 1,2,\dots,m,$

$f(u_j) = m + j + 1, \; j = 1,2,\dots,m-3,$

$f(u_{m-2}) = 2m,$

$f(u_{m-1}) = 2m-1,$

$f(u_m) = 2m+1,$

$f\left(w_j^{(i)}\right) = 2m+1+(j-1)n+i,$

$j = 1,2,\dots,n, \; i = 1,\dots,m-3,$

$f\left(w_j^{(m-2)}\right) = 2m+1+(m-2)n+j,$

$j = 1,2,\dots,n,$

$f\left(w_j^{(m-1)}\right) = 2m+1+(m-3)n+j,$

$j = 1,2,\dots,n,$

$f\left(w_j^{(m)}\right) = 2m+1+(m-1)n+j,$

$j = 1,2,\dots,n.$

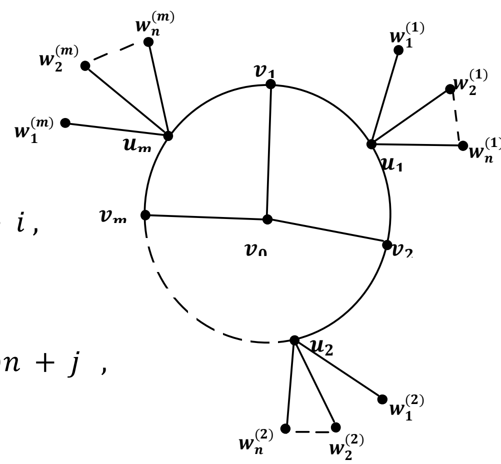


Figure (11)

Figure (12) shows a combination labeling of $G_{5,4}$.

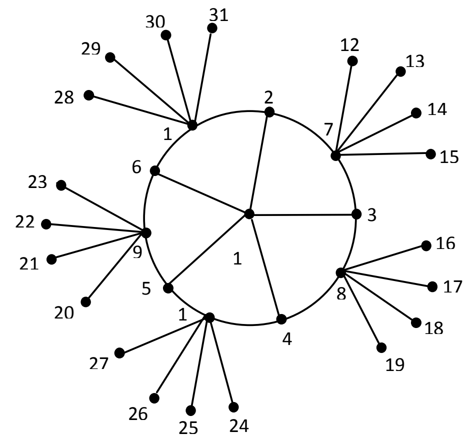


Figure (12)

Corollary 2.10.

(a) All gears are combination graphs.
(b) The graph resulting from $G_{n,m}$ after deleting the edges joining the center to the rim is a combination graph.

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