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(R, S)-Modules and their Fully and Jointly Prime Submodules

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Abstract

In this paper, the notions of (R, S)-modules and left multiplication (R, S)-modules are introduced. We also define and investigate fully prime (R, S)-submodules and jointly prime (R, S)-submodules. Characterizations of fully prime (R, S)-submodules and jointly prime (R, S)-submodules are obtained. Moreover, jointly prime (R, S)-submodules of left multiplication (R, S)-modules are classified in terms of products of (R, S)-submodules.

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1 (R, S)-Modules

Throughout this paper, let R and S be rings and M an abelian group. Bimodules over rings are well-known structures. Namely, if M is a left R-module and a right S-module and M satisfies the property $r(ms) = (rm)s$ for all $r \in R$, $m \in M$ and $s \in S$, then M can be regarded as a bimodule over R and S. For the basic properties of bimodules over rings the reader may refer to [2] and [4]. In this section, we introduce (R, S)-modules as a generalization of bimodules.

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Definition 1.1. *Let R and S be rings and M an abelian group under addition. We say that M is an (R, S)**-module** if there is a function $\_\cdot\_\cdot\_ :$ $R \times M \times S \to M$ satisfying the following properties: for all $r, r_1, r_2 \in R$, $m, n \in M$ and $s, s_1, s_2 \in S$,*

(i) $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$

(ii) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$

(iii) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$

(iv) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$.

We usually abbreviate $r \cdot m \cdot s$ by rms. We may also say that M is an (R, S)-module under $+$ and $\_\cdot\_\cdot\_$.

*An (R, S)**-submodule** of an (R, S)-module M is a subgroup N of M such that $rns \in N$ for all $r \in R$, $n \in N$ and $s \in S$.*

It is obvious that a ring R is an (R, R)-module via the usual multiplication on the ring R. Moreover, any ideals of R are (R, R)-submodules. However, an (R, R)-submodule of the (R, R)-module R need not be an ideal of the ring R.

Example 1.2. *Let $SU_4(R)$ be the ring of all 4×4 strictly upper triangular matrices over the ring R. Then $SU_4(R)$ is an $(SU_4(R), SU_4(R))$-module. Furthermore, let*

$$N = \left\{ \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid x, y \in R \right\}.$$

Then N is an $(SU_4(R), SU_4(R))$-submodule of $SU_4(R)$ since $SU_4(R)NSU_4(R) = 0$. We can see that N is not an ideal of the ring $SU_4(R)$ because it is not a left (also not a right) ideal of $SU_4(R)$ as follows:

$$SU_4(R)N = NSU_4(R) = \left\{ \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\} \not\subseteq N.$$

Note that if R is a ring with identity, then ideals of the ring R and (R, R)-submodules of the (R, R)-module R are identical.

It is easy to check that a bimodule over R and S is also an (R, S)-module. The following example shows (R, S)-modules are a true generalization of bimodules.

Example 1.3. *Let A be a ring. Then*

$$R=\left\{\begin{bmatrix} x & y \\ 0 & 0\end{bmatrix} \mid x,y\in A\right\} \quad \textit{and} \quad S=\left\{\begin{bmatrix} x & 0 \\ y & 0\end{bmatrix} \mid x,y\in A\right\}$$

are noncommutative rings without identity under the usual matrix addition and multiplication. Furthermore, let

$$M_1=\left\{\begin{bmatrix} x & 0 \\ y & z\end{bmatrix} \mid x,y,z\in A\right\} \quad \textit{and} \quad M_2=\left\{\begin{bmatrix} x & y \\ 0 & z\end{bmatrix} \mid x,y,z\in A\right\}.$$

Then M_1 and M_2 are (R,S)-modules under the usual matrix addition and multiplication. It is easy to check that M_1 is a right S-module but not a left R-module. Similarly, M_2 is a left R-module but not a right S-module. Hence M_1 and M_2 are not bimodules.

If R and S satisfy certain conditions then any (R,S)-module can be regarded as a bimodule. Recall that an element $\alpha\in R$ is called a **central idempotent** if $\alpha^2=\alpha$ and $\alpha x=x\alpha$ for all $x\in R$.

Proposition 1.4. *Let M be an (R,S)-module. If the rings R and S have central idempotents, then there are left R-module and right S-module structures on M such that $r(ms)=(rm)s$ for all $r\in R$, $m\in M$ and $s\in S$, i.e., M is a bimodule over the rings R and S.*

Proof. For each central idempotents α of R and β of S, define $\cdot_\alpha : M\times S\to M$ by $\cdot_\alpha(m,s)=\alpha ms$ for all $m\in M$ and $s\in S$, and $\cdot_\beta : R\times M\to M$ by $\cdot_\beta(r,m)=rm\beta$ for all $m\in M$ and $r\in S$. The rest of the proof is straighforward. □

Let M be an (R,S)-module. For any nonempty subsets X,Y and Z of R,M and S, respectively, we define the following sets:

$$XYZ=\{\sum_{\text{finite}} x_iy_iz_i \mid x_i\in X, y_i\in Y \text{ and } z_i\in Z \text{ for all } i\},$$
$$\langle Y\rangle=\bigcap\{\, K \mid K \text{ is an } (R,S)\text{-submodule of } M \text{ containing } Y\}\,.$$

Clearly $\langle Y\rangle$ is an (R,S)-submodule of M for any subset Y of M.

Proposition 1.5. *Let M be an (R,S)-module, X_1,X_2 be nonempty subsets of R, Y a nonempty subset of M and Z_1,Z_2 nonempty subsets of S. Then $X_1(X_2YZ_1)Z_2=(X_1X_2)Y(Z_1Z_2)$.*

Proof. This is straightforward. □

For the rest of this paper, let R and S be rings and M an (R,S)-module.

2 Fully and jointly prime submodules

Recall that a proper submodule N of a unital left R-module M is called **prime** if for each $r \in R$ and $m \in M$, $rm \in N$ implies $rM \subseteq N$ or $m \in N$. Prime submodules have been studied in many papers; see, for examples, [1], [3] and [5]. There are several possible ways to extend the definition of prime R-module to (R, S)-modules, none of which is clearly better than the others. Therefore in this paper we will introduce two such extensions, which we call fully prime and jointly prime (R, S)-modules, and study some of their properties. The current section will define and characterize fully prime and jointly prime (R, S)-modules.

The following proposition is a major tool for characterizing fully prime and jointly prime (R, S)-modules. Its proof is simple and is therefore omitted.

Proposition 2.1. *Let N be an (R, S)-submodule of M and X and Y nonempty subsets of R and S, respectively. If M satisfies $a \in RaS$ for all $a \in M$, then the following properties hold.*

(i) *(a) If $(RX)MS \subseteq N$, then $XMS \subseteq N$.*
(b) $XMS \subseteq (XR)MS$.

(ii) *(a) If $RM(YS) \subseteq N$, then $RMY \subseteq N$.*
(b) $RMY \subseteq RM(SY)$.

(iii) *$W \subseteq RWS$ for all subsets W of M. Moreover, equality holds if W is an (R, S)-submodule of M.*

We would like to point out that parts (i)*(b)* and (ii)*(b)* of Proposition 2.1 are also valid if the condition "$a \in RaS$ for all $a \in M$" is replaced by "$RMS = M$".

Now, we give the definition of fully prime (R, S)-submodules.

Definition 2.2. *A proper (R, S)-submodule P of M is called* ***fully prime*** *if for each left ideal I of R, right ideal J of S and (R, S)-submodule N of M,*

$$INJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } N \subseteq P \text{ or } RMJ \subseteq P.$$

The condition $a \in RaS$ for all $a \in M$ is crucial for proving the following characterization of fully prime (R, S)-submodules.

Theorem 2.3. *Let M be an (R, S)-module satisfying the property that $a \in RaS$ for all $a \in M$ and let P be a proper (R, S)-submodule of M. Then the following statements are equivalent.*

(i) *P is fully prime.*

(ii) *For all right ideals I of R, $m \in M$ and left ideals J of S,*

$$ImJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } m \in P \text{ or } RMJ \subseteq P.$$

(iii) *For all right ideals I of R, (R, S)-submodules N of M and left ideals J of S,*

$$INJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } N \subseteq P \text{ or } RMJ \subseteq P.$$

(iv) *For all left ideals I of R, $m \in M$ and right ideals J of S,*

$$(IR)m(SJ) \subseteq P \text{ implies } IMS \subseteq P \text{ or } m \in P \text{ or } RMJ \subseteq P.$$

(v) *For all $a \in R, m \in M$ and $b \in S$,*

$$(aR)m(Sb) \subseteq P \text{ implies } aMS \subseteq P \text{ or } m \in P \text{ or } RMb \subseteq P.$$

Proof. **(i) → (ii)** Assume (i). Let I be a right ideal of R, $m \in M$ and J a left ideal of S such that $ImJ \subseteq P$. Then $(RI)(RmS)(JS) \subseteq P$. By (i), we have $(RI)MS \subseteq P$ or $RmS \subseteq P$ or $RM(JS) \subseteq P$. It follows from Proposition 2.1 that $IMS \subseteq P$ or $m \in P$ or $RMJ \subseteq P$.

(ii) → (iii) Assume (ii). Let I be a right ideal of R, J a left ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \setminus P$. Then $InJ \subseteq INJ \subseteq P$ so that $IMS \subseteq P$ from (ii).

(iii) → (iv) Assume (iii). Let I be a left ideal of R, $m \in M$ and J a right ideal of S such that $(IR)m(SJ) \subseteq P$. Then $(IR)(RmS)(SJ) \subseteq P$. We obtain from (iii) that $(IR)MS \subseteq P$ or $RmS \subseteq P$ or $RM(SJ) \subseteq P$. By Proposition 2.1, we have $IMS \subseteq P$ or $m \in P$ or $RMJ \subseteq P$.

(iv) → (i) Assume (iv). Let I be a left ideal of R, J a right ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \setminus P$. Then $(IR)n(SJ) \subseteq P$. By (iv), we have $IMS \subseteq P$.

(ii) → (v) This is obtained from (ii) and Proposition 2.1.

(v) → (iii) Assume (v). Let I be a right ideal of R, J a left ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \setminus P$ and $b \in J$ with $RMb \nsubseteq P$. To show that $IMS \subseteq P$, let $a \in I$. Then $(aR)n(Sb) \subseteq P$. By (v), we have $aMS \subseteq P$. This implies that $IMS \subseteq P$. □

If the condition $a \in RaS$ for all $a \in M$ is replaced by $RMS = M$, then one-sided ideals of R and S can be replaced by ideals of R and S, respectively, in order to verify fully prime (R, S)-submodules.

Theorem 2.4. *Let M be an (R,S)-module satisfying $RMS = M$ and P a proper (R,S)-submodule of M. Then P is fully prime if and only if for all ideals I of R, ideals J of S and (R,S)-submodules N of M, $INJ \subseteq P$ implies $IMS \subseteq P$ or $N \subseteq P$ or $RMJ \subseteq P$.*

Proof. The direction $\Rightarrow$ follows from the definition.

For $\Leftarrow$, let I be a left ideal of R, J a right ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Then $(IR)N(SJ) \subseteq P$. Since IR is an ideal of R and SJ is an ideal of S, we have $(IR)MS \subseteq P$ or $N \subseteq P$ or $RM(SJ) \subseteq P$. If $(IR)MS \subseteq P$, then $IMS = I(RMS)S = (IR)M(SS) \subseteq (IR)MS \subseteq P$. Similarly, if $RM(SJ) \subseteq P$, then $RMJ \subseteq P$. This shows that $IMS \subseteq P$ or $N \subseteq P$ or $RMJ \subseteq P$. Hence P is a fully prime. □

At this point, we will define and study another extension of the concept of prime R-modules: jointly prime (R,S)-submodules. It will be seen that jointly prime (R,S)-submodules are generalizations of fully prime (R,S)-submodules.

Definition 2.5. *A proper (R,S)-submodule P of M is called* ***jointly prime*** *if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,*

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

It is clear that every fully prime (R,S)-submodule is a jointly prime (R,S)-submodule. Moreover, if either R or S is commutative, it can be shown that jointly prime (R,S)-submodules and fully prime (R,S)-submodules are identical.

Theorem 2.6. *Let M be an (R,S)-module. If R or S is a commutative ring, then fully prime (R,S)-submodules and jointly prime (R,S)-submodules are the same.*

Proof. It is sufficient to consider the case where R is commutative, since the case where S is commutative is nearly identical. Thus, assume R is commutative. It suffices to show that a jointly prime (R,S)-submodule is a fully prime (R,S)-submodule. Let P be a jointly prime (R,S)-submodule, I a left ideal of R, N an (R,S)-submodule of M and J a right ideal of S, such that $INJ \subseteq P$. Since P is jointly prime, $IMJ \subseteq P$ or $N \subseteq P$. Clearly we only need to consider the case $IMJ \subseteq P$. Since R is commutative, $I(RMJ)S = R(IMJ)S \subseteq P$. Again, from the fact that P is jointly prime, we obtain that $IMS \subseteq P$ or $RMJ \subseteq P$. Hence P is fully prime. □

Characterizations of jointly prime (R,S)-submodules are obtained in the same manner as those of fully prime (R,S)-submodules.

Theorem 2.7. *Let M be an (R,S)-module satisfying $a \in RaS$ for all $a \in M$ and let P be a proper (R,S)-submodule of M. The following statments are equivalent.*

(i) *P is jointly prime.*

(ii) *For all right ideals I of R, $m \in M$ and left ideals J of S,*

$$ImJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } m \in P.$$

(iii) *For all right ideals I of R, (R, S)-submodules N of M and left ideals J of S,*

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

(iv) *For all left ideals I of R, $m \in M$ and right ideals J of S,*

$$(IR)m(SJ) \subseteq P \text{ implies } IMJ \subseteq P \text{ or } m \in P.$$

(v) *For all $a \in R, m \in M$ and $b \in S$,*

$$(aR)m(Sb) \subseteq P \text{ implies } aMb \subseteq P \text{ or } m \in P.$$

Theorem 2.8. *Let M be an (R, S)-module satisfying $RMS = M$ and P a proper (R, S)-submodule of M. Then P is jointly prime if and only if for all ideals I of R, ideals J of S and (R, S)-submodules N of M, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$.*

The following propositions show that a maximal (R, S)-submodule of an (R, S)-module is always jointly prime, and is also fully prime if an additional condition is satisfied.

Proposition 2.9. *Every maximal (R, S)-submodule of an (R, S)-module is a jointly prime (R, S)-submodule.*

Proof. Let K be a maximal (R, S)-submodule of an (R, S)-module M. Let I be a left ideal of R, N an (R, S)-submodule of M and J a right ideal of S such that $INJ \subseteq K$ and $N \nsubseteq K$. Then $M = N + K$. Thus $IMJ = I(N + K)J = INJ + IKJ \subseteq K$. Hence K is a jointly prime (R, S)-submodule. □

Next example shows that jointly prime (R, S)-submodules need not be maximal.

Example 2.10. *Let $r, s \in \mathbb{Z}^+ \setminus \{1\}$. Then $\mathbb{Z}$ is an $(r\mathbb{Z}, s\mathbb{Z})$-module. Moreover, $(rs)\mathbb{Z}$ is a jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule but not maximal.*

Proposition 2.11. *Let M be an (R, S)-module such that $RMS = M$. Then every maximal (R, S)-submodule of M is a fully prime (R, S)-submodule.*

Proof. Let K be a maximal (R, S)-submodule of M. Then K is a jointly prime. We prove that K is fully prime by using Theorem 2.4. Let I be an ideal of R, N an (R, S)-submodule of M and J an ideal of S such that $INJ \subseteq K$. Assume that $N \not\subseteq K$ and $RMJ \not\subseteq K$. Then $M = RMJ + K$. Since K is jointly prime and $N \not\subseteq K$, we have $IMJ \subseteq K$. Hence

$$IMS = I(RMJ + K)S = I(RMJ)S + IKS \\ = (IR)M(JS) + IKS \subseteq IMJ + IKS \subseteq K.$$

Therefore, K is a fully prime (R, S)-submodule. □

For each (R, S)-submodule P of M, let

$$(P : M)_R = \{r \in R \mid rMS \subseteq P\}.$$

In general $(P : M)_R$ is only an additive subgroup of R. However, if $S^2 = S$ then it is easy to show that $(P : M)_R$ is an ideal of R.

Proposition 2.12. *Let P be an (R, S)-submodule of M such that $(P : M)_R$ is a proper ideal of R. If P is a jointly prime (R, S)-submodule of M, then $(P : M)_R$ is a prime ideal of R.*

In particular, if P is a fully prime (R, S)-submodule of M, then $(P : M)_R$ is a prime ideal of R.

Proof. Assume that P is a jointly prime (R, S)-submodule of M. Let A and B be ideals of R such that $AB \subseteq (P : M)_R$. We see that $A(BMS)S = (AB)M(SS) \subseteq (AB)MS \subseteq P$. Then $AMS \subseteq P$ or $BMS \subseteq P$ because P is jointly prime. Hence $A \subseteq (P : M)_R$ or $B \subseteq (P : M)_R$. This shows that $(P : M)_R$ is a prime ideal of R. □

The converse of Proposition 2.12 is invalid in general. For example, $4\mathbb{Z}$ is a $(\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. Note that $(4\mathbb{Z} : \mathbb{Z})_{\mathbb{Z}} = 2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ but $4\mathbb{Z}$ is not a jointly prime $(\mathbb{Z}, 2\mathbb{Z})$-submodule and, of course, $4\mathbb{Z}$ is not a fully prime $(\mathbb{Z}, 2\mathbb{Z})$-submodule.

3 Left multiplication (R, S)-modules

A unital left R-module M is called a **multiplication module** provided for each submodule N of M, there exists an ideal I of R such that $N = IM$ (see [3]). Some characterizations of multiplication modules are given by Z. El-Bast and P.F. Smith in [3].

Later, R. Ameri [1] defined products of submodules of a multiplication module and used them to characterize the prime submodules of a multiplication module.

Our objective is to introduce and study left multiplication (R,S)-modules. Some characterizations of left multiplication (R,S)-modules are given. Moreover, we obtain that for each (R,S)-submodule N of M, there is a unique maximal ideal J of R such that $N = JMS$. This, in fact, is a quite significant result because it allows us to define products of (R,S)-submodules.

Definition 3.1. *Let R and S be rings and M an (R,S)-module. Then M is called a* ***left multiplication*** (R,S)***-module*** *provided that for each (R,S)-submodule N of M there exists an ideal I of R such that $N = IMS$.*

The following results are analogous to the characterization of multiplication modules given by Z. El-Bast and P.F. Smith (compare with Proposition 1.1 in [3]).

Proposition 3.2. *Let M be an (R,S)-module.*

(i) *If M is a left multiplication (R,S)-module, then $RMS = M$.*

(ii) *If $RMS = M$, then $(N:M)_R$ is an ideal of R for any (R,S)-submodules N of M.*

(iii) *If M is a left multiplication (R,S)-module, then for each (R,S)-submodule N of M, $N = (N:M)_R MS$.*

(iv) *If $S^2 = S$, then M is a left multiplication (R,S)-module if and only if for each (R,S)-submodule N of M, $N = (N:M)_R MS$.*

Proof. (i) Assume that M is a left multiplication (R,S)-module. Since M is an (R,S)-submodule of M, there is an ideal I of R such that $M = IMS$. Then $M = IMS \subseteq RMS \subseteq M$, so $M = RMS$.

(ii) Assume that $RMS = M$ and let N be an (R,S)-submodule of M. It is obvious that $(N:M)_R \neq \emptyset$. Next, let $r \in R$ and $a, b \in (N:M)_R$. Then $aMS \subseteq N$ and $bMS \subseteq N$. Hence $(a-b)MS \subseteq aMS + bMS \subseteq N$,

$$\begin{aligned}(ra)MS &= (ra)(RMS)S = (raR)M(SS) = (raR)(RMS)(SS)\\ &= (raRR)M(SSS) \subseteq (raR)M(SSS) = r(aRMSS)S = r(aMS)S \subseteq N\end{aligned}$$

and

$$\begin{aligned}(ar)MS &= (ar)(RMS)S = (arR)M(SS) \subseteq (aR)M(SS)\\ &= a(RMS)S = aMS \subseteq N.\end{aligned}$$

Therefore $ar, ra \in (N:M)_R$. This implies that $(N:M)_R$ is an ideal of R.

(iii) Assume that M is a left multiplication (R,S)-module and let N be an (R,S)-submodule of M. Then $N = IMS$ for some ideal I of R. Clearly,

$I \subseteq (N : M)_R$. This implies that $N = IMS \subseteq (N : M)_R MS \subseteq N$. Hence $N = (N : M)_R MS$.

(iv) Assume that $S^2 = S$. The direction $\Rightarrow$ follows from (i). The converse holds because $S^2 = S$ implies that for any (R, S)-submodule N, $(N : M)_R$ is an ideal of R. □

We would like to emphasize that if M is a left multiplication (R, S)-module, then $M = RMS$ so that $(N : M)_R$ is an ideal of R for any (R, S)-submodule N of M.

Proposition 3.3. *Let M be an (R, S)-module. Then M is a left multiplication (R, S)-module if and only if for each $m \in M$ there exists an ideal I of R such that $\langle m \rangle = IMS$.*

Proof. The direction $\Rightarrow$ is clear. For the converse, let N be an (R, S)-submodule of M. Note that for each $n \in N$ there exists an ideal I_n of R such that $\langle n \rangle = I_n MS$. Then $I = \sum_{n \in N} I_n$ is an ideal of R satisfying $N = IMS$. □

If M is a left multiplication (R, S)-module and N is an (R, S)-submodule of M, then there may be many ideals I of R such that $N = IMS$; that is, I is not uniquely determined by N. Fortunately, we can recover uniqueness by choosing the maximal ideal J such that $N = JMS$. Note that if $N = IMS$ then $I \subseteq (N : M)_R$, and more generally, if K is an ideal such that $KMS \subseteq N$ then $K \subseteq (N : M)_R$.

Since each (R, S)-submodule of a left multiplication (R, S)-module is associated with a well-defined ideal of R, namely $(N : M)_R$, and we have concepts of primality in both cases, it is natural to ask whether the primality of one implies the primality of the other. The following result provides one answer to that question.

Theorem 3.4. *Let M be a left multiplication (R, S)-module and P an (R, S)-submodule of M. If P is a jointly prime (R, S)-submodule of M, then $(P : M)_R$ is a prime ideal of R.*

Furthermore, if R is commutative and $S^2 = S$, then the converse holds, i.e., if $(P : M)_R$ is a prime ideal of R, then P is a jointly prime (R, S)-submodule of M.

Proof. Assume that P is a jointly prime (R, S)-submodule of M. Then the ideal $(P : M)_R$ of R is proper because $M = RMS$ so that $(P : M)_R$ is a prime ideal of R by Proposition 2.12.

Conversely, assume that R is commutative, $S^2 = S$ and $(P : M)_R$ is a prime ideal of R. Let I and J be ideals of R and S, respectively, and N an (R, S)-submodule of M such that $INJ \subseteq P$. Moreover, let K and L be ideals of R such that $N = KMS$ and $IMJ = LMS$. Then

$$\begin{aligned} R(INJ)S &= (RI)KMS(JS) \\ &= (RIK)M(SJS) \\ &= (KI)RMS(JS) \\ &= (KI)M(JS) \\ &= (KL)M(SS) \\ &= (KL)MS. \end{aligned}$$

Thus $(KL)MS \subseteq P$. This implies that $KL \subseteq (P : M)_R$. Since $(P : M)_R$ is a prime ideal of R, $K \subseteq (P : M)_R$ or $L \subseteq (P : M)_R$. Hence $N \subseteq P$ or $IMJ \subseteq P$. Therefore P is a jointly prime (R, S)-submodule. □

Note that, if R is commutative, then jointly prime (R, S)-submodules and fully prime (R, S)-submodules coincide. Hence we can conclude that the converses of the two parts of Proposition 2.12 hold in a left multiplication (R, S)-module where R is commutative and $S^2 = S$.

Theorem 3.5. *Let P be an (R, S)-submodule of a left multiplication (R, S)-module M, where the ring R is commutative and $S^2 = S$. Then the following statements are equivalent.*

(i) *P is a fully prime (R, S)-submodule.*

(ii) *P is a jointly prime (R, S)-submodule.*

(iii) *$(P : M)_R$ is a prime ideal of R.*

The existence of the ideals $(N : M)_R$ allows us to define the product of two (R, S)-submodules of an arbitrary (R, S)-module.

Definition 3.6. *Let N and K be (R, S)-submodules of a left multiplication (R, S)-module M. The* ***product*** *of N and K, denoted by NK, is defined by*

$$(N : M)_R(K : M)_RMSS.$$

Clearly, the product NK is an (R, S)-submodule of M and is contained in $N \cap K$. In fact, products of (R, S)-submodules are independent of the choices of ideals of R provided R is commutative.

Proposition 3.7. *Let N and K be (R, S)-submodules of a left multiplication (R, S)-module M. If R is commutative, then $NK = (AB)M(SS)$ for any ideals A and B of R such that $N = AMS$ and $K = BMS$.*

Proof. Let A and B be ideals of R such that $(N : M)_R MS = N = AMS$ and $(K : M)_R MS = K = BMS$. Then

$$\begin{aligned} NK = [(N : M)_R (K : M)_R] M(SS) &= (N : M)_R [(K : M)_R MS] S \\ &= (N : M)_R [BMS] S \\ &= [(N : M)_R B] M(SS) \\ &= [B(N : M)_R] M(SS) \\ &= B[(N : M)_R MS] S \\ &= B(AMS)S \\ &= (BA)M(SS) \\ &= (AB)M(SS). \end{aligned}$$

This shows that $NK = (AB)M(SS)$ for any ideals A and B of R such that $N = AMS$ and $K = BMS$. □

Compare the following results with Theorem 3.16 in [1].

Proposition 3.8. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a jointly prime (R, S)-submodule, then for all (R, S)-submodules U and V of M,*

$$UV \subseteq P \quad \textit{implies} \quad U \subseteq P \textit{ or } V \subseteq P. \tag{1}$$

Furthermore, if R is commutative and $S^2 = S$, then the converse is true as well, i.e., if P satisfies condition (1), then P is a jointly prime (R, S)-submodule.

Proof. Assume that P is a jointly prime (R, S)-submodule. Let U and V be (R, S)-submodules of M such that $UV \subseteq P$. Then $(U : M)_R[(V : M)_R MS]S \subseteq P$. Since P is jointly prime, $U = (U : M)_R MS \subseteq P$ or $V = (V : M)_R MS \subseteq P$.

Conversely, assume that R is commutative, $S^2 = S$, and condition (1) holds. By Theorem 3.4, it is enough to show that $(P : M)_R$ is a prime ideal of R. Let A and B be ideals of R such that $AB \subseteq (P : M)_R$. Then $(AMS)(BMS) \subseteq P$. From condition (1) we obtain that $AMS \subseteq P$ or $BMS \subseteq P$. Hence $A \subseteq (P : M)_R$ or $B \subseteq (P : M)_R$. Therefore, $(P : M)_R$ is a prime ideal of R, as desired. □

Corollary 3.9. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a jointly prime (R, S)-submodule, then for each $a, b \in M$,*

$$\langle a \rangle \langle b \rangle \subseteq P \quad \textit{implies} \quad a \in P \quad \textit{or} \quad b \in P. \tag{2}$$

Furthermore, if R is commutative and $S^2 = S$, then the converse holds, i.e., if P satisfies condition (2), then P is a jointly prime (R, S)-submodule.

Proof. If P is a jointly prime (R, S)-submodule, then the truth of condition (2) for any $a, b \in M$ follows from Proposition 3.8.

Next, assume that R is commutative, $S^2 = S$ and condition (2) holds. Let U and V be (R, S)-submodules of M such that $UV \subseteq P$. Suppose that $U \nsubseteq P$ and $V \nsubseteq P$. Then there exist $u \in U \setminus P$ and $v \in V \setminus P$. Note that $\langle u \rangle \langle v \rangle \subseteq UV \subseteq P$, and thus condition (2) yields $u \in P$ or $v \in P$, which is a contradiction. Hence $U \subseteq P$ or $V \subseteq P$. Therefore P is a jointly prime (R, S)-submodule. □

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