International Mathematical Forum, Vol. 7, 2012, no. 35, 1699 - 1704

On Operator Inequalities Involving Numerical Radius and Operator Norm[1]

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Abstract. We prove operator inequalities involving numerical radius and norm of a bounded linear operator acting on a Hilbert space H.

Mathematics Subject Classification: Primary 47A63, Secondary 47A12

1. Introduction

Suppose T be a bounded linear operator on a complex Hilbert space H with inner product (,) and norm $\| \ \|$. Let W(T), $\sigma(T)$ denote respectively the numerical range, spectrum of T and $w(T)$, $r_\sigma(T)$ denote respectively the numerical radius, spectral radius of T, i.e.,

$$W(T) \ = \ \{(Tx, x) : \|x\| = 1\} \ and \ w(T) \ = \ \sup\{| \ \lambda \ | : \ \lambda \in W(T)\}.$$

It is easy to see that $w(T)$ is a norm on B(H), the Banach algebra of all bounded linear operators on H. Also $w(T)$ is equivalent to the usual operator norm $\|T\|$ on B(H) as

$$\frac{\|T\|}{2} \leq w(T) \leq \|T\| \ldots \ (i)$$

Kittaneh [3] substantially improved on the second inequality to prove that if T is a bounded linear operator on a complex Hilbert space H then

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} \ldots \ (ii)$$

Clearly $\frac{1}{2}\|T\|+\frac{1}{2}\|T^2\|^{\frac{1}{2}} \leq \|T\|$ so that inequality (ii) is sharper than the second inequality of (i). The significant part in inequality (ii) is the contribution made by the second factor involving $\|T^2\|$. Some easy examples mentioned

[1]The research of first author is partially supported by PURSE-DST, Govt. of India and research of second author is supported by CSIR, India.

below illustrate the fact that one can not compare $w(T)$ with $\|T^2\|^{\frac{1}{2}}$. If T is a 2×2 nilpotent matrix with index 2 then one may get $w(T) = \frac{1}{2}$ and $\|T^2\| = 0$ whereas if T is a 3×3 nilpotent matrix with index 3 then one may get $w(T) = \frac{1}{\sqrt{2}}$ and $\|T^2\| = 1$. U.Haagerup and P.De La Harpe [2] estimated the numerical radius of a nilpotent operator on a Hilbert space and proved that

$$w(T) \leq \|T\| \cos\frac{\pi}{n+1},\ where\ T^n = 0\ for\ some\ n \geq 2,$$

the equality holds when T is the n-dimensional shift on the space C^n.
Let $T = U \mid T \mid$ be the polar decomposition of T, then the Aluthge [1] transform $\tilde{T}$ of T is defined as $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$. Using the inequality (ii) of Kittaneh, T.Yamazaki [5] obtained an inequality concerning operator norm $\|T\|$, numerical radius $w(T)$ and Aluthge transform $\tilde{T}$ of T as follows

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\tilde{T}).$$

Letting $\tilde{T}_0 = T$ and $\tilde{T}_n = \tilde{\tilde{T}}_{n-1}$ for natural number n, Yamazaki also proved that

$$w(T) \leq \sum_{n=1}^{\infty} \frac{1}{2^n}\|\tilde{T}_{n-1}\|.$$

Recently in [4] we proved the following theorem using Archimedean Property

Theorem 1. *Let T be a bounded linear operator on a complex Hilbert space H. Then either there exists some $n_0 \in N$ such that*

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}} \tag{1}$$

or for all $n \in N$

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T). \tag{2}$$

2. Main results

Using the Archimedean property we here first prove two operator inequalities either of which has to be true.

Theorem 2.1. *Let T be a bounded linear operator on a complex Hilbert space H. Then either there exists some $n_1 \in N$ such that*

$$\|T\| \leq w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}} \tag{3}$$

or for all $n \in N$

$$w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < \|T\| \tag{4}$$

Proof. We know that $\frac{\|T\|}{2} \leq w(T) \leq \|T\|$. We first note that if $w(T) = \|T\|$ then it satisfies the inequality (3).
If $\|T\| > w(T)$ then by Archimedean property there exists $n \in N$ such that

$$n(\|T\| - w(T)) > \frac{1}{2^2}\|T^2\|^{\frac{1}{2}}.$$

So

$$\|T\| > w(T) + \frac{1}{2^2 n}\|T^2\|^{\frac{1}{2}}.$$

Let $S = \{n \in N : \|T\| > w(T) + \frac{1}{2^2 n}\|T^2\|^{\frac{1}{2}}\}$. Then $S \neq \phi$ and so S has a least element $k_1 \in N$. If $k_1 \neq 1$ then

$$w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} < \|T\| \leq w(T) + \frac{1}{2^2(k_1 - 1)}\|T^2\|^{\frac{1}{2}}$$

or if $k_1 = 1$ then we get

$$\|T\| > w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}}.$$

In both cases i.e., for $k_1 > 1$ and $k_1 = 1$ we have $\|T\| > w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}}$.
Again by Archimedean property there exists $n \in N$ such that

$$n(\|T\| - w(T) - \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}}) > \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}.$$

As before we can find a least element $k_2 \in N$ such that if $k_2 \neq 1$ then

$$w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3 k_2}\|T^{2^2}\|^{\frac{1}{2^2}} < \|T\| \leq w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3(k_2 - 1)}\|T^{2^2}\|^{\frac{1}{2^2}}$$

or if $k_2 = 1$ then

$$\|T\| > w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}.$$

Proceeding in this way we get a sequence of natural numbers $\{k_n\}$ such that either of the following two cases arise
Case 1. $k_n \neq 1$ *for some* n. In this case

$$w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n k_{n-1}}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \frac{1}{2^{n+1} k_n}\|T^{2^n}\|^{\frac{1}{2^n}} < \|T\|$$

and

$$\|T\| \leq w(T) + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n k_{n-1}}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \frac{1}{2^{n+1}(k_n - 1)}\|T^{2^n}\|^{\frac{1}{2^n}}.$$

This is a new operator inequality involving both lower and upper bounds of numerical radius.
Case 2. $k_n = 1\ \forall n \in N$. In this case for all $n \in N$

$$w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < \|T\|.$$

This is a new operator inequality involving upper bound of the numerical radius.
If Case 1 holds we get the existence of $n_1 \in N$ such that

$$\|T\| \leq w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}}$$

and if Case 2 holds then we get $\forall n \in N$

$$w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < \|T\|.$$

This completes the proof.

Remark 2.2. In the proof of theorem if $k_1 = 2$ then

$$w(T) + \frac{1}{2^2.2}\|T^2\|^{\frac{1}{2}} < \|T\| \leq w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} < w(T) + \frac{1}{2}\|T^2\|^{\frac{1}{2}}$$

so that

$$\frac{1}{2^2.2}\|T^2\|^{\frac{1}{2}} < \|T\| - w(T) \leq \frac{1}{2^2}\|T^2\|^{\frac{1}{2}}$$

If $k_1 = 1, k_2 = 2$ then

$$w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3.2}\|T^{2^2}\|^{\frac{1}{2^2}} < \|T\| \leq w(T) + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}$$

so that

$$\frac{5}{16}\|T^{2^2}\|^{\frac{1}{2^2}} < \|T\| - w(T) \leq \frac{3}{8}\|T^2\|^{\frac{1}{2}}.$$

Remark 2.3. From **Theorem 1** and **Theorem 2.1** we conclude that for any bounded linear operator T
(*) either the inequality (1) or (2) holds and
(**) either the inequality (3) or (4) holds.

We now prove the following theorem

Theorem 2.4. *Suppose T be a bounded linear operator on a complex Hilbert space H. Then one of the following four alternatives is true*

$$\|T\| < c\|T^2\|^{1/2},\ for\ some\ constant\ c \in [1, 2)$$

$$w(T) < \frac{3}{4}\|T\|,\ w(T) > \frac{3}{4}\|T\|,\ r_\sigma(T) \leq \frac{1}{2}\|T\|,$$

Proof. We have from Theorem 1, either there exists $n_0 \in N$ such that

$$(1) \qquad w(T) \leq \tfrac{1}{2}\|T\| + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}}$$

$$or\ \forall n \in N\ (2) \qquad \tfrac{1}{2}\|T\| + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T).$$

We also have from Theorem 2.1, either there exists $n_1 \in N$ such that

$$(3) \qquad \|T\| \leq w(T) + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n_1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}}$$

$$or\ \forall n \in N\ (4) \qquad w(T) + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < \|T\|.$$

Now we investigate the following four options.

Case 1. When (1) and (3) holds.

Without loss of generality we assume that $n_0 \leq n_1$. Then we get

$$\begin{aligned} w(T) + \|T\| &\leq \frac{1}{2}\|T\| + w(T) + 2[\frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}}] \\ &\quad + \frac{1}{2^{n_0+1}}\|T^{2^{n_0}}\|^{\frac{1}{2^{n_0}}} + \ldots + \frac{1}{2^{n_1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}} \\ \Rightarrow \|T\| &\leq \|T^2\|^{\frac{1}{2}} + \frac{1}{2}\|T^{2^2}\|^{\frac{1}{2^2}} + \ldots + \frac{1}{2^{n_0-2}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}} \\ &\quad + \frac{1}{2^{n_0}}\|T^{2^{n_0}}\|^{\frac{1}{2^{n_0}}} + \ldots + \frac{1}{2^{n_1-1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}} \\ \Rightarrow \|T\| &\leq [(1 - \frac{1}{2^{n_1-1}}) + (1 - \frac{1}{2^{n_0-1}})]\|T^2\|^{1/2} \end{aligned}$$

Thus we get $\|T\| \leq [(1 - \frac{1}{2^{n_1-1}}) + (1 - \frac{1}{2^{n_0-1}})]\|T^2\|^{1/2}$ where $n_1, n_0 \in N$ and $n_1 \geq 2, n_0 \geq 2$. Hence we conclude

$$\|T\| \leq c\|T^2\|^{1/2}\ for\ some\ constant\ c \in [1, 2).$$

If $n_1 = n_0 = 2$ then $\|T\| \leq \|T^2\|^{1/2}$ and so $\|T\| = \|T^2\|^{1/2}$ as we know $\|T^2\|^{1/2} \leq \|T\|$.

Case 2. When (1) and (4) holds.

As (4) holds for all $n \in N$ so it holds for n_0 and adding (1) and (4) we get

$$\begin{aligned} & w(T) + w(T) + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}} \\ < \ & \tfrac{1}{2}\|T\| + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}} + \|T\| \\ \Rightarrow \ & w(T) < \tfrac{3}{4}\|T\|. \end{aligned}$$

Case 3. When (2) and (3) holds.

As (2) holds for all $n \in N$ so it holds for n_1 and adding (2) and (3) we get

$$\begin{aligned} & \tfrac{1}{2}\|T\| + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \|T\| \\ & < w(T) + w(T) + \tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n_1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}} \\ \Rightarrow \ & w(T) > \tfrac{3}{4}\|T\|. \end{aligned}$$

Case 4. When (2) and (4) holds.

Adding (2) and (4) we get for all $n \in N$

$$\tfrac{1}{2}\|T\| + w(T) + 2[\tfrac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}}] < w(T) + \|T\|$$

$$\Rightarrow \tfrac{1}{2}\|T^2\|^{\frac{1}{2}} + \ldots + \tfrac{1}{2^{n-1}}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < \tfrac{1}{2}\|T\|.$$

Thus

$$(1-\frac{1}{2^n})\|T^{2^n}\|^{\frac{1}{2^n}} < \frac{1}{2}\|T\| \; for\ all\ n \in N.$$

Taking limit as $n \longrightarrow \infty$ we get $r_\sigma(T) \leq \frac{1}{2}\|T\|$.
This completes the proof.

Corollary 2.5. Let T be a bounded linear operator on H. Then one of the following four inequalities holds.

$$\|T\| < \|T^2\|^{1/2} + c\|T^{2^2}\|^{\frac{1}{2^2}}, \; for\ some\ c \in [0,1)$$

$$or\ \frac{1}{2}\|T\| \leq w(T) < \frac{3}{4}\|T\| \ or\ \frac{3}{4}\|T\| < w(T) \leq \|T\| \ or\ r_\sigma(T) \leq \frac{1}{2}\|T\|.$$

Proof. As in the last theorem we have from Case 1,

$$\begin{aligned} \|T\| &\leq \|T^2\|^{\frac{1}{2}} + \frac{1}{2}\|T^{2^2}\|^{\frac{1}{2^2}} + \ldots + \frac{1}{2^{n_0-2}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}} \\ &\quad + \frac{1}{2^{n_0}}\|T^{2^{n_0}}\|^{\frac{1}{2^{n_0}}} + \ldots + \frac{1}{2^{n_1-1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}} \\ \Rightarrow \|T\| &\leq \|T^2\|^{\frac{1}{2}} + [1-(\frac{1}{2^{n_1-1}} + \frac{1}{2^{n_0-1}})]\|T^{2^2}\|^{1/2^2} \end{aligned}$$

where $n_1, n_0 \in N$ and $n_1 \geq 2, n_0 \geq 2$. Thus

$$\|T\| < \|T^2\|^{1/2} + c\|T^{2^2}\|^{\frac{1}{2^2}}, \; for\ some\ c \in [0,1).$$

Remaining inequalities follow from the other three cases of the last theorem.

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Received: January, 2012