On Operator Inequalities Involving Numerical Radius and Operator Norm

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Abstract. We prove operator inequalities involving numerical radius and norm of a bounded linear operator acting on a Hilbert space H.

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1. Introduction

Suppose T be a bounded linear operator on a complex Hilbert space H with inner product ( , ) and norm || ||. Let W(T), σ(T) denote respectively the numerical range, spectrum of T and w(T), rσ(T) denote respectively the numerical radius, spectral radius of T, i.e.,

\[ W(T) = \{(Tx, x) : \|x\| = 1\} \text{ and } w(T) = \sup\{|\lambda| : \lambda \in W(T)\}. \]

It is easy to see that w(T) is a norm on B(H), the Banach algebra of all bounded linear operators on H. Also w(T) is equivalent to the usual operator norm \|T\| on B(H) as

\[ \frac{\|T\|}{2} \leq w(T) \leq \|T\| \ldots (i) \]

Kittaneh [3] substantially improved on the second inequality to prove that if T is a bounded linear operator on a complex Hilbert space H then

\[ w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^\frac{1}{2} \ldots (ii) \]

Clearly \( \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^\frac{1}{2} \leq \|T\| \) so that inequality (ii) is sharper than the second inequality of (i). The significant part in inequality (ii) is the contribution made by the second factor involving \( \|T^2\| \). Some easy examples mentioned

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below illustrate the fact that one can not compare \( w(T) \) with \( \|T^2\|^{\frac{1}{2}} \). If \( T \) is a \( 2 \times 2 \) nilpotent matrix with index 2 then one may get \( w(T) = \frac{1}{2} \) and \( \|T^2\| = 0 \) whereas if \( T \) is a \( 3 \times 3 \) nilpotent matrix with index 3 then one may get \( w(T) = \frac{1}{2} \) and \( \|T^2\| = 1 \). U.Haagerup and P.De La Harpe [2] estimated the numerical radius of a nilpotent operator on a Hilbert space and proved that

\[
w(T) \leq \|T\| \cos \frac{\pi}{n+1}, \text{ where } T^n = 0 \text{ for some } n \geq 2,
\]

the equality holds when \( T \) is the \( n \)-dimensional shift on the space \( C^n \).

Let \( T = U \mid T \mid \) be the polar decomposition of \( T \), then the Aluthge [1] transform \( \tilde{T} \) of \( T \) is defined as \( \tilde{T} = |T|^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}} \). Using the inequality (ii) of Kittaneh, T.Yamazaki [5] obtained an inequality concerning operator norm \( \|T\| \), numerical radius \( w(T) \) and Aluthge transform \( \tilde{T} \) of \( T \) as follows

\[
w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} w(\tilde{T}).
\]

Letting \( \tilde{T}_0 = T \) and \( \tilde{T}_n = \tilde{T}_{n-1} \) for natural number \( n \), Yamazaki also proved that

\[
w(T) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|\tilde{T}_{n-1}\|.
\]

Recently in [4] we proved the following theorem using Archimedean Property

**Theorem 1.** Let \( T \) be a bounded linear operator on a complex Hilbert space \( H \). Then either there exists some \( n_0 \in N \) such that

\[
(1) \quad w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n} \|T^{2^n-1}\| \frac{1}{2^{n-1}}
\]

or for all \( n \in N \)

\[
(2) \quad \frac{1}{2} \|T\| + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n} \|T^{2^n-1}\| \frac{1}{2^{n-1}} < w(T).
\]

**2. Main results**

Using the Archimedean property we here first prove two operator inequalities either of which has to be true.

**Theorem 2.1.** Let \( T \) be a bounded linear operator on a complex Hilbert space \( H \). Then either there exists some \( n_1 \in N \) such that

\[
(3) \quad \|T\| \leq w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\| \frac{1}{2^{n_1-1}}
\]

or for all \( n \in N \)

\[
(4) \quad w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n} \|T^{2^n-1}\| \frac{1}{2^{n-1}} < \|T\|
\]
Proof. We know that $\|T\|/2 \leq w(T) \leq \|T\|$. We first note that if $w(T) = \|T\|$ then it satisfies the inequality (3).
If $\|T\| > w(T)$ then by Archimedean property there exists $n \in N$ such that
\[
n(\|T\| - w(T)) > \frac{1}{2^2} \|T^2\|^{1/2}.
\]
So
\[
\|T\| > w(T) + \frac{1}{2^n} \|T^2\|^{1/2}.
\]
Let $S = \{n \in N : \|T\| > w(T) + \frac{1}{2^n} \|T^2\|^{1/2}\}$. Then $S \neq \emptyset$ and so S has a least element $k_1 \in N$. If $k_1 \neq 1$ then
\[
w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} < \|T\| \leq w(T) + \frac{1}{2^2(k_1 - 1)} \|T^2\|^{1/2}
\]
or if $k_1 = 1$ then we get
\[
\|T\| > w(T) + \frac{1}{2^2} \|T^2\|^{1/2}.
\]
In both cases i.e., for $k_1 > 1$ and $k_1 = 1$ we have $\|T\| > w(T) + \frac{1}{2k_1} \|T^2\|^{1/2}$.
Again by Archimedean property there exists $n \in N$ such that
\[
n(\|T\| - w(T) - \frac{1}{2^2k_1} \|T^2\|^{1/2}) > \frac{1}{2^3} \|T^2\|^{1/2^n}.
\]
As before we can find a least element $k_2 \in N$ such that if $k_2 \neq 1$ then
\[
w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} + \frac{1}{2^3k_2}\|T^2\|^{1/2^n} < \|T\| \leq w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} + \frac{1}{2^3(k_2 - 1)} \|T^2\|^{1/2^n}
\]
or if $k_2 = 1$ then
\[
\|T\| > w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} + \frac{1}{2^3}\|T^2\|^{1/2^n}.
\]
Proceeding in this way we get a sequence of natural numbers $\{k_n\}$ such that either of the following two cases arise

**Case 1.** $k_n \neq 1$ for some $n$. In this case
\[
w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} + \ldots + \frac{1}{2^n k_{n-1}} \|T^{2^{n-1}}\|^{1/2^n} + \frac{1}{2^{n+1}k_n} \|T^{2^n}\|^{1/2^n} < \|T\|
\]
and
\[
\|T\| \leq w(T) + \frac{1}{2^2k_1}\|T^2\|^{1/2} + \ldots + \frac{1}{2^n k_{n-1}} \|T^{2^{n-1}}\|^{1/2^n} + \frac{1}{2^{n+1}(k_n - 1)} \|T^{2^n}\|^{1/2^n}.
\]
This is a new operator inequality involving both lower and upper bounds of numerical radius.

**Case 2.** $k_n = 1$ for all $n \in N$. In this case for all $n \in N$
\[
w(T) + \frac{1}{2^2}\|T^2\|^{1/2} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{1/2^n} < \|T\|.
\]
Remark 2.3. In the proof of theorem if $k_1 = 2$ then

$$w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} < \|T\| \leq w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} < \|T\| + \frac{1}{2} \|T^2\|^{\frac{1}{2}}$$

so that

$$\frac{1}{2^2} \|T^2\|^{\frac{1}{2}} < \|T\| - w(T) \leq \frac{1}{2^2} \|T^2\|^{\frac{1}{2}}$$

If $k_1 = 1, k_2 = 2$ then

$$w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \frac{1}{2^3} \|T^2\|^{\frac{3}{2}} < \|T\| \leq w(T) + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \frac{1}{2^3} \|T^2\|^{\frac{3}{2}}$$

so that

$$\frac{5}{16} \|T^2\|^{\frac{3}{2}} < \|T\| - w(T) \leq \frac{3}{8} \|T^2\|^{\frac{1}{2}}.$$

Remark 2.2. From Theorem 1 and Theorem 2.1 we conclude that for any bounded linear operator $T$

(*) either the inequality (1) or (2) holds and

(**) either the inequality (3) or (4) holds.

We now prove the following theorem

**Theorem 2.4.** Suppose $T$ be a bounded linear operator on a complex Hilbert space $H$. Then one of the following four alternatives is true

$$\|T\| < c\|T^2\|^{1/2}, \text{ for some constant } c \in [1, 2)$$

$$w(T) < \frac{3}{4} \|T\|, \ w(T) > \frac{3}{4} \|T\|, \ r_\sigma(T) \leq \frac{1}{2} \|T\|,$$

**Proof.** We have from Theorem 1, either there exists $n_0 \in N$ such that

$$w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}} \|T^{2^{n_0}-1}\|^{\frac{1}{2^{n_0}-1}}$$

or $\forall n \in N$

$$\frac{1}{2} \|T\| + \frac{1}{2^2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n} \|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T).$$
We also have from Theorem 2.1, either there exists \( n_1 \in \mathbb{N} \) such that

\[
\tag{3} \|T\| \leq w(T) + \frac{1}{2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}}
\]

or for all \( n \in \mathbb{N} \)

\[
\tag{4} w(T) + \frac{1}{2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n} \|T^{2^n-1}\|^{\frac{1}{2^n-1}} < \|T\|.
\]

Now we investigate the following four options.

**Case 1.** When (1) and (3) holds.
Without loss of generality we assume that \( n_0 \leq n_1 \). Then we get

\[
w(T) + \|T\| \leq \frac{1}{2} \|T\| + w(T) + 2\left[\frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}} \|T^{2^{n_0}-1}\|^{\frac{1}{2^{n_0}-1}}\right]
\]

\[
= \|T\| + 2\left[\frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}} \|T^{2^{n_0}-1}\|^{\frac{1}{2^{n_0}-1}}\right] + \frac{1}{2^{n_0+1}} \|T^{2^{n_0}}\|^{\frac{1}{2^{n_0+1}}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}}
\]

\[
\Rightarrow \|T\| \leq \left[1 - \frac{1}{2^{n_1-1}}\right] + \left[1 - \frac{1}{2^{n_0-1}}\right] \|T^2\|^{1/2}
\]

Thus we get \( \|T\| \leq \left[1 - \frac{1}{2^{n_1-1}}\right] + \left[1 - \frac{1}{2^{n_0-1}}\right] \|T^2\|^{1/2} \) where \( n_1, n_0 \in \mathbb{N} \) and \( n_1 \geq 2, n_0 \geq 2 \). Hence we conclude

\[
\|T\| \leq c \|T^2\|^{1/2} \text{ for some constant } c \in [1, 2).
\]

If \( n_1 = n_0 = 2 \) then \( \|T\| \leq \|T^2\|^{1/2} \) and so \( \|T\| = \|T^2\|^{1/2} \) as we know \( \|T^2\|^{1/2} \leq \|T\| \).

**Case 2.** When (1) and (4) holds.
As (4) holds for all \( n \in \mathbb{N} \) so it holds for \( n_0 \) and adding (1) and (4) we get

\[
w(T) + \|T\| < w(T) + \frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}} \|T^{2^{n_0}-1}\|^{\frac{1}{2^{n_0}-1}} + \|T\|
\]

\[
\Rightarrow w(T) < \frac{3}{4} \|T\|.
\]

**Case 3.** When (2) and (3) holds.
As (2) holds for all \( n \in \mathbb{N} \) so it holds for \( n_1 \) and adding (2) and (3) we get

\[
\frac{1}{2} \|T\| + \frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}} + \|T\|
\]

\[
< w(T) + \frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}} + \|T\|
\]

\[
\Rightarrow w(T) > \frac{3}{4} \|T\|.
\]

**Case 4.** When (2) and (4) holds.
Adding (2) and (4) we get for all \( n \in \mathbb{N} \)

\[
\frac{1}{2} \|T\| + w(T) + 2\left[\frac{1}{2^n} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}}\right] < w(T) + \|T\|
\]

\[
\Rightarrow \frac{1}{2} \|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_1}} \|T^{2^{n_1}-1}\|^{\frac{1}{2^{n_1}-1}} < \frac{1}{2} \|T\|.
\]
Thus
\[(1 - \frac{1}{2^n})\|T^{2n}\|^{\frac{1}{2n}} < \frac{1}{2}\|T\| \text{ for all } n \in \mathbb{N}.
\]
Taking limit as \(n \to \infty\) we get \(r_\sigma(T) \leq \frac{1}{2}\|T\|.
This completes the proof.

**Corollary 2.5.** Let \(T\) be a bounded linear operator on \(H\). Then one of the following four inequalities holds.

\[\|T\| < \|T^2\|^{1/2} + c\|T^{2^2}\|^{\frac{1}{2^2}}, \text{ for some } c \in [0, 1)\]

\[\text{or } \frac{1}{2}\|T\| \leq w(T) < \frac{3}{4}\|T\| \text{ or } \frac{3}{4}\|T\| < w(T) \leq \|T\| \text{ or } r_\sigma(T) \leq \frac{1}{2}\|T\|.
\]

**Proof.** As in the last theorem we have from Case 1,
\[
\|T\| \leq \|T^2\|^{\frac{1}{2}} + \frac{1}{2}\|T^{2^2}\|^{\frac{1}{2^2}} + \ldots + \frac{1}{2^{n_0-2}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}}
\]
\[
+ \frac{1}{2^{n_0}}\|T^{2^{n_0}}\|^{\frac{1}{2^{n_0}}} + \ldots + \frac{1}{2^{n_1-1}}\|T^{2^{n_1-1}}\|^{\frac{1}{2^{n_1-1}}}
\]
\[
\Rightarrow \|T\| \leq \|T^2\|^{\frac{1}{2}} + [1 - (\frac{1}{2^{n_1-1}} + \frac{1}{2^{n_0-1}})]\|T^{2^2}\|^{1/2^2}
\]
where \(n_1, n_0 \in \mathbb{N}\) and \(n_1 \geq 2, n_0 \geq 2\). Thus
\[
\|T\| < \|T^2\|^{1/2} + c\|T^{2^2}\|^{\frac{1}{2^2}}, \text{ for some } c \in [0, 1).
\]
Remaining inequalities follow from the other three cases of the last theorem.

**References**


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