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On the Regularity of Distributions under Pure Order Differential Operators based on the Continuous Wavelet Transform

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Abstract. Given a distribution solution u in $\mathcal{E}'(\mathbb{R}^n)$ for a non-homogeneous equation under a differential operator of pure order, the regularity of u is studied by considering the regularity of the continuous wavelet transform of u which is defined with respect to a radially symmetric admissible function.

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1 Introduction

The regularity of distributions under different kind of differential operators has been studied for a long time ago, see for example [2], where locally solvable differential operators are used. Also, the global, local, and pointwise regularity of distributions with compact support by using wavelet frames has been studied in [5]. Moreover, but by using the continuous wavelet transform, the regular points for distributions u with compact support, that is in $\mathcal{E}'(\mathbb{R}^n)$ has been studied recently, see [3].

The analysis of regularity for distributions u with compact support, based on its continuous wavelet transform is a technique that may be applied for partial differential operators with C^∞ coefficients.

Thus, in this paper, for a given a distribution u in $\mathcal{E}'(\mathbb{R}^n)$, we define the continuous wavelet transform of u with respect to an admissible function h in $L^2(\mathbb{R}^n)$ so that if v in $\mathcal{E}'(\mathbb{R}^n)$ is given, and u satisfies the equation $Qu = v$ where $Q = \sum_{|\alpha|=p} \partial^p$ is a differential operator of pure order p, then for v in $C^\infty(\mathbb{R}^n)$, it follows that u is also of class $C^\infty(\mathbb{R}^n)$. We should say that this result is an extension to the study of regularity of functions f and u in $L^2(\mathbb{R}^n)$, where $Qu = f$, see [4].

Most of the support of these ideas are based on the $L^2(\mathbb{R}^n)$ results given in [3].

2 Notations and definitions

Let us begin by defining a group action on $L^2(\mathbb{R}^n)$. So, we have the following definition concerning the translation and dilation operators.

Definition 1 *For h in $L^2(\mathbb{R}^n)$ and x in $\mathbb{R}^n$, we define the following operators.*

$$\begin{aligned} (T_b h)(x) &= h(x-b), \quad b \in \mathbb{R}^n \\ (J_a h)(x) &= a^{-\frac{n}{2}}\, h\left(\frac{x}{a}\right), \quad a > 0 \end{aligned} \tag{2.1}$$

Definition 2 *Consider now the following set*

$$G = \{(a,b)|\, a > 0,\ \ b \in \mathbb{R}^n\}. \tag{2.2}$$

Then in G define the following product: $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$.

Remark 1 *With this product G becomes a locally compact topological group, where the identity is $(1, 0)$ and the inverse is $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$. The affine group G is not unimodular, as the left and right invariant measures $d(a, b) = a^{-n-1}\, db\, da$ and $d_r(a, b) = a^{-1}\, db\, da$ differ.*

Definition 3 *For (a, b) in G, define the two-parameter family of operators*

$$U(a,b) = T_b J_a \tag{2.3}$$

Remark 2 *The family of operators $U(a, b) = T_b J_a$ is a representation of G acting on $L^2(\mathbb{R}^n)$ by:*

$$U(a,b)h(x) = T_b J_a h(x) = J_a h(x-b) = a^{-\frac{n}{2}}\, h\left(\frac{x-b}{a}\right). \tag{2.4}$$

Then we have the following definition for admissible functions.

Definition 4 *A function h in $L^2(\mathbb{R}^n)$ is said to be admissible if*

$$\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) < \infty. \tag{2.5}$$

Hence, the admissibility condition for radially symmetric functions h in $L^2(\mathbb{R}^n)$, see [1] is:

$$C_h \equiv \int_0^\infty |\eta(k)|^2 \frac{1}{k} dk < \infty, \tag{2.6}$$

where $\widehat{h}(y) = \eta(|y|)$, and where $\widehat{h}$ is the Fourier transform of h.

Definition 5 *Given $f \in L^2(\mathbb{R}^n)$ and $(a,b) \in G$, the wavelet transform of f with respect to an admissible function $h \in L^2(\mathbb{R}^n)$ is defined as:*

$$(L_h f)(a,b) = \langle f, U(a,b)h\rangle. \tag{2.7}$$

Remark 3 *Note that from Remark 2,*

$$(L_h f)(a,b) = \langle f, T_b J_a h\rangle$$

$$= \int_{\mathbb{R}^n} f(x) \frac{1}{a^{\frac{n}{2}}} \overline{h}\left(\frac{x-b}{a}\right) dx = \left[f * (J_a \overline{h})^{\sim}\right](b),$$

where $*$ means convolution and $\psi^{\sim}(x) = \psi(-x)$.

Remark 4 *For f, h in $L^2(\mathbb{R}^n)$ where h is radially symmetric and admissible, the resolution of the identity in the weak sense, see [1], is given by:*

$$f = \frac{1}{C_h} \int_G \langle f, U(a,b)h\rangle\, U(a,b)h \; d(a,b). \tag{2.8}$$

That is, for any g in $L^2(\mathbb{R}^n)$,

$$\langle f, g\rangle = \frac{1}{C_h} \int_G \langle f, U(a,b)h\rangle\, \overline{\langle g, U(a,b)h\rangle}\; d(a,b). \tag{2.9}$$

Definition 6 *Given u in $\mathcal{E}'(\mathbb{R}^n)$ and an admissible function h in $\mathcal{D}(\mathbb{R}^n)$, the wavelet transform of u with respect to h is defined as*

$$(L_h u)(a,b) = u\left[\overline{T_b J_a h}\right]. \tag{2.10}$$

3 Partial Results

In this section we establish some results about the derivative of the continuous wavelet transform for distributions. For this purpose, we will apply the following three results given in [3], (they are respectively Corollary 2, Lemma 3, and Theorem 2.)

Lemma 1 *Suppose $h \in \mathcal{D}(\mathbb{R}^n)$ is admissible and radially symmetric, and $C_h = \int_0^\infty |\eta(k)|^2 \frac{1}{k} dk < \infty$, where $\widehat{h}(x) = \eta(|x|)$. Then for any $u \in \mathcal{E}'(\mathbb{R}^n)$ and any multi-index $\gamma \in \mathbb{R}^n$,*

$$\partial_b^\gamma (L_h u)(a,b) = \frac{(-1)^{|\gamma|}}{a^{|\gamma|}} u\left[\overline{T_b J_a \partial^\gamma h}\right]. \tag{3.1}$$

Lemma 2 *If u is in $\mathcal{E}'(\mathbb{R}^n)$, then there is a multi-index $\alpha \in \mathbb{R}^n$ and compactly supported continuous functions g_λ with $\lambda \leq \alpha$, where supp $u \subset$ supp g_λ such that*

$$u[\phi] = \sum_{\lambda \leq \alpha} \int_{\mathbb{R}^n} g_\lambda(x) \partial^\lambda \phi(x)\, dx \tag{3.2}$$

for all $\phi \in \mathcal{E}(\mathbb{R}^n)$.

Lemma 3 *Suppose that $h \in \mathcal{D}(\mathbb{R}^n)$ is a non-zero admissible function that is radially symmetric. Let u be in $\mathcal{E}'(\mathbb{R}^n)$ and for (a,b) in G and any multi-index β in $\mathbb{R}^n$, let*

$$(\mathcal{W}_h^\beta u)(a,b) = a^{-\frac{2+n}{2}} \partial_b^\beta (L_h u)(a,b). \tag{3.3}$$

Then

1) $\mathcal{W}_h^\beta u$ is continuous at any point (a_1, b_1) in G for any multi-index $\beta \in \mathbb{R}^n$.

2) u is C^∞ in a neighborhood of $x = b_0$ if and only if for each multi-index $\beta \in \mathbb{R}^n$,

$$\lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^\beta u)(a,b) \quad \text{exists for each } b_1 \text{ in a neighborhood of } b_0.$$

With the help of the previous Lemmas we have the following results concerning the derivative with respect to b of the continuous wavelet transform for distributions with compact support.

Lemma 4 *Given u in $\mathcal{E}'(\mathbb{R}^n)$ and any multi-index γ in $\mathbb{R}^n$, if h in $\mathcal{D}(\mathbb{R}^n)$ is admissible, then*

$$\partial_b^\gamma (L_h u)(a,b) = (L_h \partial^\gamma u)(a,b).$$

Proof. On one hand, from (3.1) and (2.10),

$$\partial_b^\gamma (L_h u)(a,b) = \frac{(-1)^{|\gamma|}}{a^{|\gamma|}} u \left[\overline{T_b J_a \partial^\gamma h}\right] = \frac{(-1)^{|\gamma|}}{a^{|\gamma|}} (L_{\partial^\gamma h} u)(a,b).$$

On the other hand, from (2.10) and by Lemma 2, there is a multi-index α in $\mathbb{R}^n$ and compactly supported continuous functions g_λ, where $supp\, u \subset supp\, g_\lambda$ with $\lambda \leq \alpha$ so that

$$\begin{aligned}
&(L_h \partial^\gamma u)(a,b) = (\partial^\gamma u) \left[\overline{T_b J_a h}\right] = (-1)^{|\gamma|} u \left[\overline{\partial^\gamma T_b J_a h}\right] \\
&= (-1)^{|\gamma|} \sum_{\lambda \leq \alpha} \int_{\mathbb{R}^n} g_\lambda(x) \partial_x^\lambda \partial_x^\gamma (T_b J_a h)(x)\, dx \\
&= (-1)^{|\gamma|} \sum_{\lambda \leq \alpha} \int_{\mathbb{R}^n} g_\lambda(x) \partial_x^\lambda \frac{1}{a^{|\gamma|}} (T_b J_a \partial^\gamma h)(x)\, dx \\
&= \frac{(-1)^{|\gamma|}}{a^{|\gamma|}} u \left[\overline{T_b J_a \partial^\gamma h}\right] \\
&= \frac{(-1)^{|\gamma|}}{a^{|\gamma|}} (L_{\partial^\gamma h} u)(a,b).
\end{aligned}$$

This proves Lemma 4. □

Corollary 1 *Consider h in $\mathcal{D}(\mathbb{R}^n)$ such that for any multi-index α in $\mathbb{R}^n$, the function $\partial^\gamma h$ is admissible. If u is in $\mathcal{E}'(\mathbb{R}^n)$ and $Q_b = \sum_{|\alpha|=p} \partial_b^\alpha$, then*

$$Q_b (L_h u)(a,b) = \sum_{|\alpha|=p} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} u)(a,b).$$

Proof. Note that since

$$Q_b (L_h u)(a,b) = \sum_{|\alpha|=p} \partial_b^\alpha (L_h u)(a,b),$$

it follows from (3.1) and (2.10),

$$\sum_{|\alpha|=p} \partial_b^\alpha (L_h u)(a,b) = \sum_{|\alpha|=p} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} u[\overline{T_b J_a \partial^\alpha h}] = \sum_{|\alpha|=p} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} u)(a,b).$$

Thus,

$$Q_b(L_h u)(a,b) = \sum_{|\alpha|=p} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} u)(a,b).$$

This proves Corollary 1. □

Corollary 2 *Let h be in $\mathcal{D}(\mathbb{R}^n)$ such that h is admissible. If u and Qu are in $L^2(\mathbb{R}^n)$ where $Q = \sum_{|\alpha|=p} \partial^\alpha$, then*

$$Q_b(L_h u)(a,b) = (L_h Q u)(a,b). \tag{3.4}$$

Proof. On one hand, from Lemma 4, and because of the linear property of the continuous wavelet transform,

$$\begin{aligned} Q_b(L_h u)(a,b) &= \sum_{|\alpha|=p} \partial_b^\alpha (L_h u)(a,b) = \sum_{|\alpha|=p} (L_h \partial^\alpha u)(a,b) \\ &= \left(L_h \sum_{|\alpha|=p} \partial^\alpha u \right)(a,b) = (L_h Q u)(a,b). \end{aligned}$$

This proves Corollary 2. □

Lemma 5 *Suppose h in $\mathcal{D}(\mathbb{R}^n)$ is radially symmetric and admissible, where $h(0) = 0$ and $h \neq 0$. Suppose also that Ω is a domain in $\mathbb{R}^n$, and consider u, v in $\mathcal{E}'(\mathbb{R}^n)$. If v is of class C^∞ in an open neighborhood of $x = b_0$ in Ω and u is a distribution solution of $\partial^\alpha u = v$ in Ω for any multi-index $\alpha \in \mathbb{R}^n$, then u is of class C^∞ in an open neighborhood of $x = b_0$ in Ω.*

Proof. Since v in $\mathcal{E}'(\mathbb{R}^n)$ is of class C^∞ in an open neighborhood of $x = b_0$ in Ω, it follows from Lemma 3, that for any multi-index $\beta \in \mathbb{R}^n$,

$\lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^\beta v)(a,b)$ exists for any b_1 in an open neighborhood of b_0, where

$$(\mathcal{W}_h^\beta v)(a,b) = a^{-\frac{2+n}{2}} \partial_b^\beta (L_h v)(a,b).$$

But since $\partial^\alpha u = v$, it follows from Lemma 4,

$$\lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^\beta v)(a,b) = \lim_{(a,b)\to(0,b_1)} a^{-\frac{2+n}{2}} \partial_b^\beta (L_h \partial^\alpha u)(a,b)$$

$$= \lim_{(a,b)\to(0,b_1)} a^{-\frac{2+n}{2}} \partial_b^{\beta+\alpha} (L_h u)(a,b) = \lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^{\beta+\alpha} u)(a,b).$$

This means that for any multi-index $\beta \in \mathbb{R}^n$ we have the existence of the limit of $(\mathcal{W}_h^{\beta+\alpha} u)(a,b)$ as $(a,b) \to (0,b_1)$ for each b_1 in an open neighborhood of $x = b_0$ in Ω. Thus, by Lemma 3, it follows that u in $\mathcal{E}'(\mathbb{R}^n)$ is of class C^∞ in an open neighborhood of $x = b_0$.

This proves Lemma 5. □

4 Main Result

We now give the main result of this paper.

Theorem 1 *Suppose Ω is a domain in $\mathbb{R}^n$ and consider u, v in $\mathcal{E}'(\mathbb{R}^n)$. Suppose also that u is a distribution solution of $Qu = v$, where $Q = \sum_{|\alpha|=p} \partial^\alpha$ and p is a positive integer. Then for h in $L^2(\mathbb{R}^n)$ admissible and radially symmetric, if v is of class C^∞ in an open neighborhood of $b_0 \in \Omega$, then u is of class C^∞ in an open neighborhood of $b_0 \in \Omega$.*

Proof. Since v in $\mathcal{E}'(\mathbb{R}^n)$ is C^∞ in an open neighborhood of $x = b_0$ in Ω, it follows from Lemma 3, that for any multi-index $\beta \in \mathbb{R}^n$, we have the existence of the limit of $(\mathcal{W}_h^\beta v)(a,b)$ as $(a,b) \to (0,b_1)$ for any b_1 in an open neighborhood of $x = b_0$ in Ω.

Now, since

$$(\mathcal{W}_h^\beta v)(a,b) = a^{-\frac{2+n}{2}} \partial_b^\beta (L_h v)(a,b)$$

$$= a^{-\frac{2+n}{2}} \partial_b^\beta (L_h Q u)(a,b),$$

it follows from (3.4), Corollary 1, and (3.1) respectively that

$$
\begin{aligned}
(\mathcal{W}_h^{\beta} v)(a,b) &= a^{-\frac{2+n}{2}} \partial_b^{\beta} (L_h Qu)(a,b) \\
&= a^{-\frac{2+n}{2}} \partial_b^{\beta} Q_b (L_h u)(a,b) \\
&= a^{-\frac{2+n}{2}} \partial_b^{\beta} \left(\sum_{|\alpha|=p} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} u)(a,b) \right) \\
&= \sum_{|\alpha|=p} a^{-\frac{2+n}{2}} \partial_b^{\beta} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} u)(a,b) \\
&= \sum_{|\alpha|=p} a^{-\frac{2+n}{2}} \partial_b^{\beta} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} u[\overline{T_b J_a \partial^\alpha h}] \\
&= \sum_{|\alpha|=p} a^{-\frac{2+n}{2}} \partial_b^{\beta} \partial_b^{\alpha} (L_h u)(a,b) \\
&= \sum_{|\alpha|=p} a^{-\frac{2+n}{2}} \partial_b^{\beta+\alpha} (L_h u)(a,b).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^{\beta} v)(a,b) &= \sum_{|\alpha|=p} \lim_{(a,b)\to(0,b_1)} a^{-\frac{2+n}{2}} \partial_b^{\beta+\alpha} (L_h u)(a,b) \\
&= \sum_{|\alpha|=p} \lim_{(a,b)\to(0,b_1)} (\mathcal{W}_h^{\beta+\alpha} u)(a,b).
\end{aligned}
$$

This means that for any multi-index $\beta \in \mathbb{R}^n$, we have the existence of the limit of $(\mathcal{W}_h^{\beta+\alpha} u)(a,b)$ as $(a,b) \to (0,b_1)$ for each b_1 in an open neighborhood of $x = b_0$. Then from Lemma 3, we have that u in $\mathcal{E}'(\mathbb{R}^n)$ is of class C^∞ in an open neighborhood of b_0 in Ω.

This proves Theorem 1. □

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