International Mathematical Forum, Vol. 7, 2012, no. 35, 1737 - 1749

Special Classes of Divisor Cordial Graphs

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Abstract

A divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper, we proved some special classes of graphs such as dragon, corona, wheel, full binary trees, $G * K_{2,n}$ and $G * K_{3,n}$ are divisor cordial.

Mathematics Subject Classification: 05C78

Keywords: Cordial labeling, divisor cordial labeling, divisor cordial graphs

1 Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [5].

First we give the some concepts in Number Theory [3].

Definition 1.1. *Let a and b be two integers. If a divides b means that there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$.*

If a does not divide b, then we denote $a \nmid b$.

Graph labeling [4] is a strong communication between Number theory [3] and structure of graphs [5]. By combining the divisibility concept in Number theory and Cordial labeling concept in Graph labeling, we introduced a new concept called divisor cordial labeling [8]. In paper[8], we proved the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial and complete graph is not divisor cordial. In this paper we are going to prove some special classes of graphs such as full binary trees, dragon, corona, wheel, $G * K_{2,n}$ and $G * K_{3,n}$ are divisor cordial.

A vertex labeling [4] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. The two best known labeling methods are called graceful and harmonious labelings. Cordial labeling is a variation of both graceful and harmonious labelings [1].

Definition 1.2. *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

The concept of cordial labeling was introduced by Cahit [1].

Definition 1.3. *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is cordial if it admits cordial labeling.*

Cahit proved some results in [2].

2 Main Results

Sundaram, Ponraj and Somasundaram [6] have introduced the notion of prime cordial labeling.

Definition 2.1. *[6] A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1.*

In [6], they have proved some graphs are prime cordial.

Motivated by the concept of prime cordial labeling, we introduced a new special type of cordial labeling called divisor cordial labeling as follows.

Definition 2.2. *[8] Let $G=(V,E)$ be a simple graph and $f : V \to \{1, 2, \ldots |V|\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u) \nmid f(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.*

A graph with a divisor cordial labeling is called a divisor cordial graph.

Example 2.3. *Consider the following graph G.*

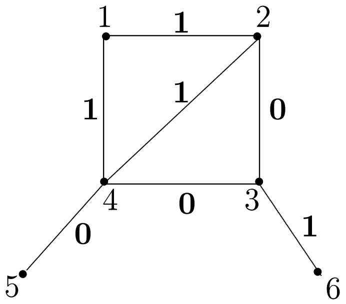

Fig 1

We see that $e_f(0) = 3$ and $e_f(1) = 4$.

Thus $|e_f(0) - e_f(1)| = 1$ and hence G is divisor cordial.

In this paper we prove some special classes of graphs are divisor cordial.

Theorem 2.4. *Given a positive integer n, there is a divisor cordial graph G which has n vertices.*

Proof. Suppose n is even.

Construct a path containing $\frac{n}{2}+2$ vertices $v_1, v_2, \ldots, v_{\frac{n}{2}+2}$ which are labeled as $1, 2, \ldots, \frac{n}{2}+2$ respectively. Note that the label of the edge v_1v_2 is 1 and the other edges v_iv_{i+1} $(2 \leq i \leq \frac{n}{2}+1)$ have the labels 0.

Attach $\frac{n}{2}-2$ vertices $v_{\frac{n}{2}+3}, v_{\frac{n}{2}+4}, \ldots, v_n$ which are labeled as $\frac{n}{2}+3, \ldots n$ respectively, to the vertex v_1. We see that the labels of the edges $v_1v_i(\frac{n}{2}+3 \leq i \leq n)$ are 1.

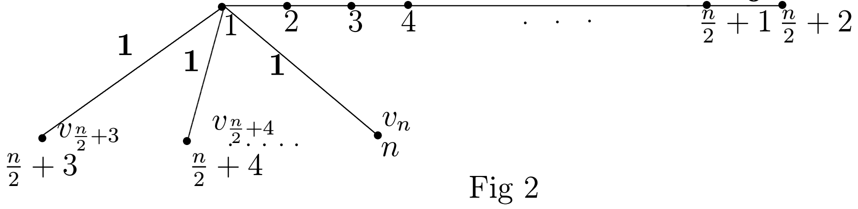

Fig 2

So, we have $e_f(0) = \frac{n}{2}$ and $e_f(1) = 1 + \frac{n}{2} - 2 = \frac{n}{2} - 1$ and hence $|e_f(0) - e_f(1)| \leq 1$. Thus, the resultant graph G is divisor cordial.,

Similarly, we can construct a graph for n is odd. □

Theorem 2.5. *If G is a divisor cordial graph of even size, then $G - e$ is also divisor cordial for all $e \in E(G)$.*

Proof. Let q be the even size of the divisor cordial graph G. Then it follows that $e_f(0) = e_f(1) = \frac{q}{2}$. Let e be any edge in G which is labeled either 0 or 1. Then in $G - e$, we have either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$ and hence $|e_f(0) - e_f(1)| \leq 1$. Thus $G - e$ is divisor cordial. □

Theorem 2.6. *If G is a divisor cordial graph of odd size, then $G - e$ is also divisor cordial for some $e \in E(G)$.*

Proof. Let q be the odd size of the divisor cordial graph G. Then it follows that either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$. If $e_f(0) = e_f(1) + 1$ then remove the edge e which is labeled as 0 and if $e_f(1) = e_f(0) + 1$ then remove the edge e which is labeled 1 from G. Then it follows that $e_f(0) = e_f(1)$. Thus $G - e$ is divisor cordial for some $e \in E(G)$. □

Note 2.7. *Similarly we can prove for the divisor cordiality of $G + e$.*

Definition 2.8. *An ordered rooted tree is a binary tree if each vertex has atmost two children.*

Definition 2.9. *A full binary tree is a binary tree inwhich each internal vertex has exactly two children.*

Next we will prove that a full binary tree is divisor cordial.

Theorem 2.10. *Every full binary tree is divisor cordial.*

Proof. We note that every full binary tree has odd number of vertices and hence has even number of edges.

Let T be a full binary tree and let v be a root of T which is called zero level vertex. Clearly, the i^{th} level of T has 2^i vertices. If T has m levels, then the number of vertices of T is $2^{m+1} - 1$ and the number of edges is 2^{m+1}.

Now assign the label 2 to the root v and assign the labels 3 and 1 to the first level vertices. Next, we assign the labels $2^i, 2^i + 1, \ldots, 2^{i+1} - 1$ to the i^{th} level vertices for $2 \leqslant i \leqslant m$

In zeroth, first and second levels, we have $e_f(0) = e_f(1) = 3$. Clearly,

$2^i + j \mid 2^{i+1} + 2j$ for $0 \leqslant j \leqslant 2^i - 2$ and

$2^i + j \nmid 2^{i+1} + 2j - 1$ for $1 \leqslant j \leqslant 2^i - 1$. Then after the second levels, we have $e_f(0) = e_f(1) = 2^m - 3$.

Thus $|e_f(0) - e_f(1)| = 0$ and hence T is divisor cordial. □

The next example shows that the divisor cordiality of full binary tree upto 4 levels.

Example 2.11. *Here we see that* $e_f(0) = e_f(1) = 15$.

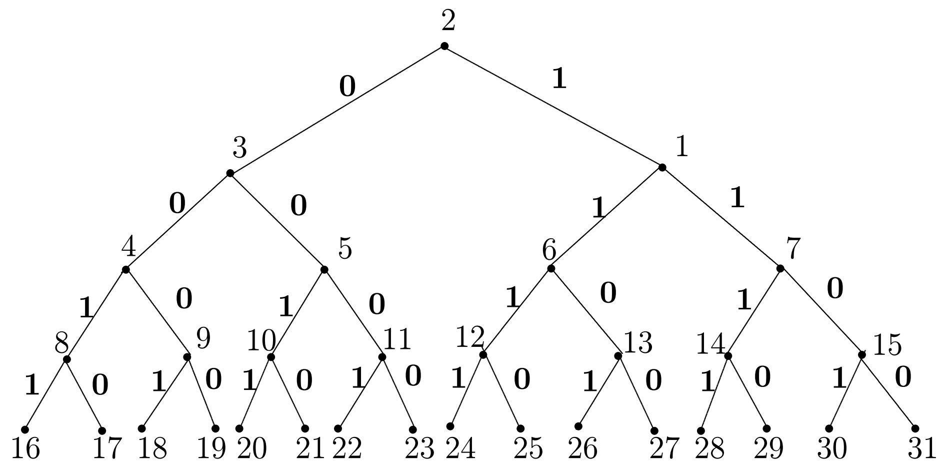

Fig 3

Theorem 2.12. *Let* G *be any divisor cordial graph of size* m *and* $K_{2,n}$ *be a bipartite graph with the bipartition* $V = V_1 \cup V_2$ *with* $V_1 = \{x_1, x_2\}$ *and* $V_2 = \{y_1, y_2, \dots y_n\}$. *Then the graph* $G * K_{2,n}$ *obtained by identifying the vertices* x_1 *and* x_2 *of* $K_{2,n}$ *with that labeled 1 and the largest prime number* p *such that* $p \le m$ *respectively in* G *is also divisor cordial.*

Proof. Let G be any divisor cordial graph of size m. Let v_k and v_l be the vertices having the labels 1 and the largest prime number namely p such that $p \le m$.

Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2, \dots y_n\}$. Now assign the labels $m+1, m+2, \dots m+n$ to the vertices $y_1, y_2, \dots, y_n$ respectively.

Now identify the vertices x_1 and x_2 of $K_{2,n}$ with that labeled 1 and the largest prime number p such that $p \le m$ respectively in G.

Case (i) $p < m + n < 2p$

Then the multiples of p are not available in the labels of y_i $(1 \le i \le n)$. Then we see that the edges of $K_{2,n}$ incident with the vertex v_k have the label 1 and with the vertex v_l have the label 0. Thus the edges of $K_{2,n}$ contribute equal numbers namely n to both $e_f(1)$ and $e_f(0)$ in $G * K_{2,n}$. Hence $G * K_{2,n}$ is divisor cordial.

Case (ii) $m + n \ge 2p$.

Let q be the largest prime number such that $q \leq m+n$ which is labeled to some y_i. Then interchange the labels of v_l and y_i, that is p and q. We observe that the largest prime number q does not divide the labels of $y_1, y_2, \ldots, y_n$. So, again the edges of $K_{2,n}$ incident with the vertex v_l have the labels 0 and hence $G * K_{2,n}$ is divisor cordial. □

The following example explains the construction developed in the Theorem 2.12.

Example 2.13. *Consider the following graph G.*

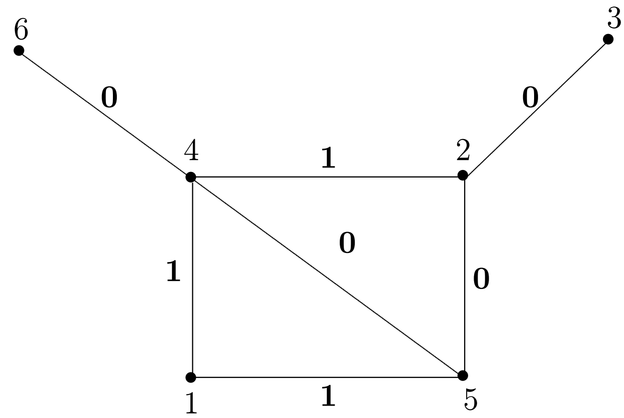

Fig 4

Here $e_f(0) = 4$ and $e_f(1) = 3$ and hence $|e_f(0) - e_f(1)| = 1$. Thus G is divisor cordial.

Case(i): Now consider the bipartite graph $K_{2,3}$. Then the graph $G * K_{2,3}$ and its labels are gives as follows.

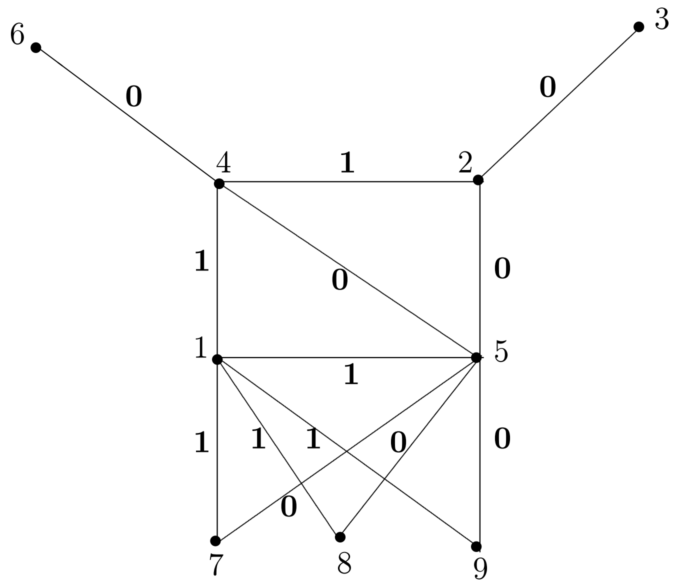

Fig 5

Here $e_f(0) = 7$ *and* $e_f(1) = 6$ *and* $|e_f(0) - e_f(1)|$ *=1. Thus* $G*K_{2,3}$ *is divisor cordial.*

Case (ii): Now consider the bipartite graph $K_{2,5}$*. The graph* $G * K_{2,5}$ *and its labels are given as follows.*

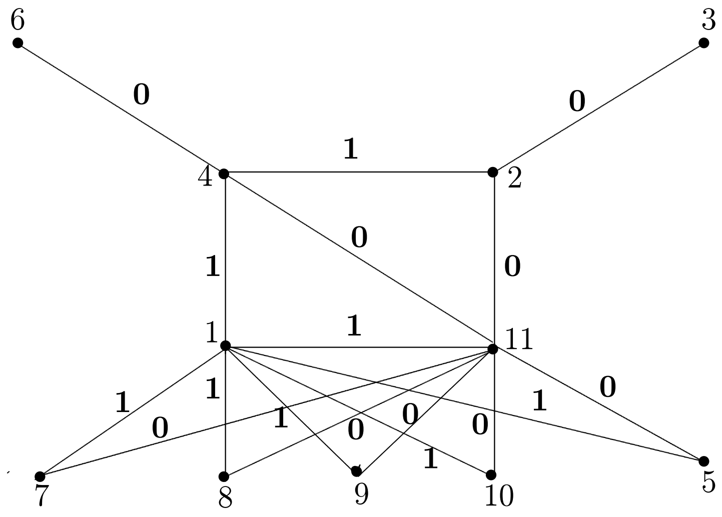

Fig 6

Here we interchanged the labels 5 and 11 and $e_f(0) = 9$ *and* $e_f(1)$ *=8 and so* $|e_f(0) - e_f(1)|$ *=1*

Thus $G * K_{2,5}$ *is divisor cordial.*

Theorem 2.14. *Let* G *be any divisor cordial graph of size* m *and* $K_{3,n}$ *be a bipartite graph with the bipartition* $V = V_1 \cup V_2$ *with* $V_1 = \{x_1, x_2, x_3\}$ *and* $V_2 = \{y_1, y_2, \ldots y_n\}$, *where* n *is even. Then the graph* $G * K_{3,n}$ *obtained by identifying the vertices* x_1, x_2 *and* x_3 *of* $K_{3,n}$ *with that labeled 1, labeled 2 and the largest prime number* p *such that* $p \leq m$ *respectively in* G *is also divisor cordial.*

Proof. Let G be any divisor cordial graph of size m. Then $|e_f(0) - e_f(1)| \leq 1$. Let v_k, v_l and v_r be the vertices having the labels 1, 2 and the largest prime number namely p such that $p \leq m$.

Let $V = V_1 \cup V_2$ be the bipartition of $K_{3,n}$ such that $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, \ldots y_n\}$. Now assign the labels $m+1, m+2, \ldots m+n$ to the vertices $y_1, y_2, \ldots, y_n$ respectively.

Now identify the vertices x_1, x_2 and x_3 of $K_{3,n}$ with that labeled 1, labeled 2 and and the largest prime number p such that $p \leq m$ respectively in G.

Case (i) $p < m + n < 2p$

Then the multiples of p are not available in the labels of y_i $(1 \leq i \leq n)$. Then we see that the edges of $K_{3,n}$ incident with the vertex v_k have the label 1 and with the vertex v_r have the label 0. Since n divides only even numbers, $\frac{n}{2}$ edges of $K_{3,n}$ incident with the vertex v_l have the label 1 and remaining $\frac{n}{2}$ edges have the label 0. Thus the edges of $K_{3,n}$ contribute equal numbers

namely $\frac{3n}{2}$ to both $e_f(1)$ and $e_f(0)$ in $G * K_{3,n}$. Hence $G * K_{3,n}$ is divisor cordial.
Case (ii) $m + n \geq 2p$.

Let q be the largest prime number such that $q \leq m + n$ which is labeled to some y_i. Then interchange the labels of v_r and y_i, that is p and q. We observe that the largest prime number q does not divide the labels of $y_1, y_2, \ldots, y_n$. So, again the edges of $K_{3,n}$ incident with the vertex v_r have the labels 0 and hence $G * K_{3,n}$ is divisor cordial. □

Note 2.15. *Theorem 2.14 is also valid for m is even and n is odd. If both m and n are odd, the above labeling pattern is not valid.*

The following example illustrates the construction developed in the Theorem 2.14.

Example 2.16. *Consider the following divisor cordial graph G.*

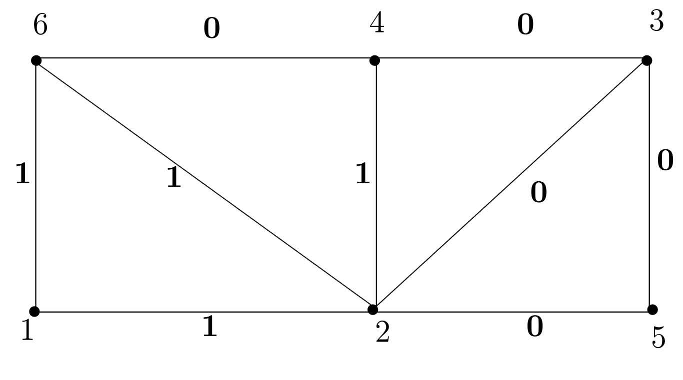

Fig 7

The divisor cordiality of $G * K_{3,4}$ is shown below.

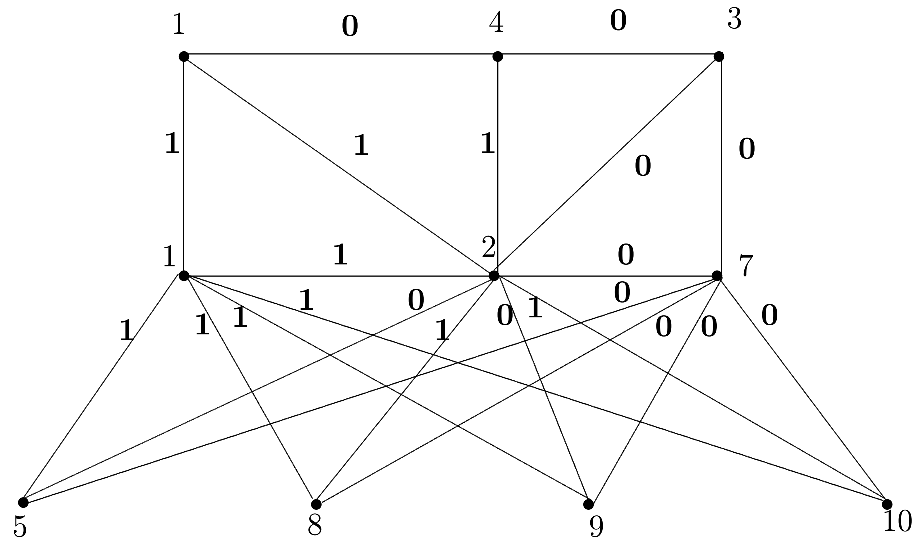

Fig 8

Here $e_f(0) = 11$ $e_f(1) = 10$.

Now we present divisor cordial labeling for the graphs obtained by joining apex vertices of two stars to a new vertex. We extend this result for three copies of stars.

Definition 2.17. *Consider two stars* $K_{1,n}^{(1)}$ *and* $K_{1,n}^{(2)}$. *Then* $G = < K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ *is the graph obtained by joining apex (central) vertices of stars to a new vertex* x.

Note that G *has* $2n+3$ *vertices and* $2n+2$ *edges.*

Definition 2.18. *Consider* t *copies of stars namely* $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(t)}$. *Then* $G = < K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(t)} >$ *is the graph obtained by joining apex vertices of each* $K_{1,n}^{(m-1)}$ *and* $K_{1,n}^{(m)}$ *to a new vertex* x_{m-1} *where* $2 \leq m \leq t$.

Note that G *has* $t(n+2)-1$ *vertices and* $t(n+2)-2$ *edges.*

First we will prove that $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ *is divisor cordial.*

Theorem 2.19. *The graph* $G = < K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ *is divisor cordial.*

Proof. Let $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and they are adjacent to a new common vertex x.

Now assign the label 1 to c_1 and the largest prime number p such that $p \leq 2n+3$ to c_2 and the remaining labels to the vertices of G. Since 1 divides any integer the edges incident to c_1 contribute $n+1$ to $e_f(1)$ and since p does not divide any labels of the vertices adjacent to c_2, the edges incident to c_2 also contribute $n+1$ to $e_f(0)$. Hence $|e_f(0)-e_f(1)|=0$. Thus G is divisor cordial. □

Next we will extend this result to 3 stars as follows.

Theorem 2.20. *The graph* $G=<K_{1,n}^{(1)},\ K_{1,n}^{(2)},\ K_{1,n}^{(3)}>$ *is divisor cordial.*

Proof. Let Let $v_1^{(i)}, v_2^{(i)}, \ldots, v_n^{(i)}$ be the pendant vertices of $K_{1,n}^{(i)}$ and let c_i be the apex vertex of $K_{1,n}^{(i)}$ for $i=1,2,3$. Now c_1 and c_2 are adjacent to x_1 and c_2 and c_3 are adjacent to x_2. Note that G has $3n+5$ vertices and $3n+4$ edges.

Now assign the label 1 to c_1, 2 to c_2 and p to c_3 where p is the largest prime number such that $p \leq 3n+5$. Then assign the odd and even labels equally to the vertices $v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}, x_1, x_2$ if n is even. Suppose n is odd, then assign $\frac{n+1}{2}$ odd labels and $\frac{n+3}{2}$ even labels to these vertices. Then assign remaining labels to the remaining vertices.

Since 1 divides any integer, the edges incident to c_1 contribute $n+1$ to $e_f(1)$. Since p does not divide any labels of the vertices adjacent to c_3, the edges incident to c_3 also contribute $n+1$ to $e_f(0)$. Since 2 divides even labels and does not divide odd labels, the edges incident to c_2 contribute $\frac{n+2}{2}$ $(\frac{n+1}{2})$ to $e_f(0)$ and $\frac{n+2}{2}$ $(\frac{n+3}{2})$ to $e_f(1)$ if n is even (odd). Thus, if n is even, then $e_f(0)=e_f(1)=n+1+\frac{n+2}{2}=\frac{3n+4}{2}$ and if n is odd then $e_f(0)=n+1+\frac{n+1}{2}$=2n+1 and $e_f(1)=n+1+\frac{n+3}{2}=2n+2$ and hence $|e_f(0)-e_f(1)| \leq 1$. Thus G is divisor cordial. □

The Theorems 2.19 and 2.20 are explained using the following example.

Example 2.21. *Consider the graph* $G=<K_{1,8}^{(1)},\ K_{1,8}^{(2)}>$.

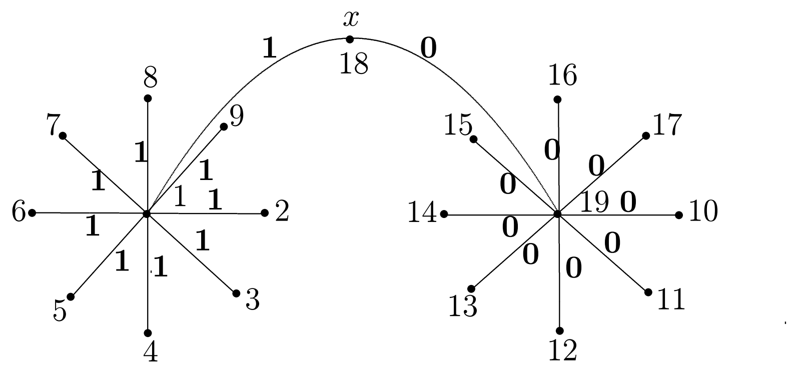

Fig 9

Here $e_f(0) = e_f(1) = 9$.

Example 2.22. *Consider the graph* $G = < K_{1,7}^{(1)}, K_{1,7}^{(2)}, K_{1,7}^{(3)} >$.

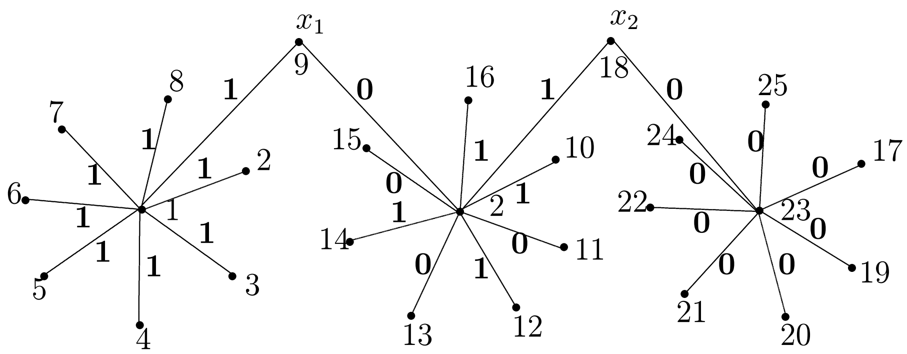

Fig 10

Here $e_f(0) = 12$ *and* $e_f(1) = 13$ *and* $|e_f(0) - e_f(1)| = 1$.

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Received: November, 2011