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A Note on the Uniqueness of Involution in Real Locally C*-Algebras

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Abstract. In the present note we show that the involution in real locally C*-algebras is uniquely determined.

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1. Introduction

One of the important basic facts of the theory of C^***-algebras** is that the unary operation of involution in a C^*-algebra is uniquely determined. This property was first observed in 1955 by Bohnenblust and Karlin in [2] (see as well [15] for a nice exposition). This result was extended in 2000 to real C^*-algebras by Z.Y. Li (see [12]).

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [1] and Michael [14]. The Hausdorff projective limits of projective families of C^*-algebras were first mentioned by Arens [1]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [3]. We will follow Inoue [5] in the usage of the name **locally** C^***-algebras** for these algebras. Real analogues of locally C^*-algebras- so called **real locally** C^***-algebras** as projective limits of projective families of real C^*-algebras were first introduced in 2006 by Katz and Friedman (see [6]).

The purpose of the present notes is to extend the aforementioned result of Z.Y. Li from [12] to real locally C^*-algebras. Otherwise speaking, it will be shown here that the unary operation of involution in real locally C^*-algebras is uniquely determined.

2. Preliminaries

First, we recall some basic notions on topological *-algebras. A *-algebra (or involutive algebra) is an algebra B over $\mathbb{C}$ with an involution

$$^* : B \to B,$$

such that

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^*,$$

and

$$(ab)^* = b^* a^*,$$

for every $a, b \in B$ and $\lambda \in \mathbb{C}$.

A seminorm $\|.\|$ on a *-algebra B is a C^*-seminorm if it is submultiplicative, i.e.

$$\|ab\| \leq \|a\| \, \|b\| ,$$

and satisfies the C^*-condition, i.e.

$$\|a^* a\| = \|a\|^2 ,$$

for every $a, b \in B$. Note that the C^*-condition alone implies that $\|.\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\| ,$$

for every $a \in B$ (cf. for example [3]).

When a seminorm $\|.\|$ on a *-algebra B is a C^*-norm, and A is complete in in the topology generated by this norm, B is called a **C^*-algebra**. The following theorem is valid.

Theorem 1 (Bohnenblust and Karlin [2])**.** *The unary operation of involution in a C^*-algebra is uniquely determined.*

Proof. See for example [15] for details. □

A topological *-algebra is a *-algebra B equipped with a topology making the operations (addition, multiplication, additive inverse, involution) jointly continuous. For a topological *-algebra B, one puts $N(B)$ for the set of continuous C^*-seminorms on B. One can see that $N(B)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|.\|_\alpha , \|.\|_\beta\} \in N(B)$$

for every $\|.\|_\alpha , \|.\|_\beta \in N(B)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological *-algebra B, and $\|.\|_\alpha \in N(B)$, $\alpha \in \Lambda$,

$$\ker \|.\|_\alpha = \{a \in B : \|a\|_\alpha = 0\}$$

is a *-ideal in B, and $\|.\|_\alpha$ induces a C^*-norm (we as well denote it by $\|.\|_\alpha$) on the quotient $B_\alpha = B/\ker \|.\|_\alpha$, and B_α is automatically complete in the topology generated by the norm $\|.\|_\alpha$, thus is a C^*-algebra (see [3] for details). Each pair $\|.\|_\alpha, \|.\|_\beta \in N(B)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective *-homomorphism

$$g_\alpha^\beta : B_\beta \to B_\alpha.$$

Let, again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " $\preceq$ ". Let

$$\{B_\alpha, \alpha \in \Lambda\}$$

be a family of C^*-algebras, and g_α^β be, for

$$\alpha \preceq \beta,$$

the continuous linear *-mappings

$$g_\alpha^\beta : B_\beta \longrightarrow B_\alpha,$$

so that

$$g_\alpha^\alpha(x_\alpha) = x_\alpha,$$

for all $\alpha \in \Lambda$, and

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma,$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let B be a *-subalgebra of the direct product algebra

$$\prod_{\alpha \in \Lambda} B_\alpha,$$

so that for its elements

$$x_\alpha = g_\alpha^\beta(x_\beta),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_\alpha \in B_\alpha,$$

and

$$x_\beta \in B_\beta.$$

Definition 1. *The* **-algebra* B *constructed above is called a* ***Hausdorff projective limit*** *of the projective family*

$$\{B_\alpha, \alpha \in \Lambda\},$$

relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\varprojlim B_\alpha,$$

and is called the Arens-Michael decomposition of B*.*

It is well known (see, for example [16]) that for each $x \in B$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : B \longrightarrow B_\beta,$$

defined by

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

Definition 2. *A topological* **-algebra* B *over* $\mathbb{C}$ *is called a* ***locally*** C^****-algebra*** *if there exists a projective family of* C^**-algebras*

$$\{B_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$B \cong \varprojlim B_\alpha,$$

i.e. A *is topologically* **-isomorphic to a projective limit of a projective family of* C^**-algebras, i.e. there exits its Arens-Michael decomposition of* A *composed entirely of* C^**-algebras.*

A topological *-algebra B over $\mathbb{C}$ is a locally C^*-algebra iff B is a complete Hausdorff topological *-algebra in which topology is generated by a saturated separating family of C^*-seminorms (see [3] for details).

Example 1. *Every* C^**-algebra is a locally* C^**-algebra.*

Example 2. *A closed* **-subalgebra of a locally* C^**-algebra is a locally* C^**-algebra.*

Example 3. *The product* $\prod_{\alpha \in \Lambda} B_\alpha$ *of* C^**-algebras* B_α*, with the product topology, is a locally* C^**-algebra.*

Example 4. *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X, with the topology of uniform convergence on compact subsets, is a locally C^*-algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see* [11] *for details).*

Let B be a locally C^*-algebra. Then an element $a \in B$ is called **bounded**, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|.\|_\alpha \in N(B)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(B)$.

It is well-known that for each locally C^*-algebra B, its set $b(B)$ of bounded elements of B is a locally C^*-subalgebra, which is a C^*-algebra in the norm $\|.\|_\infty$, such that it is dense in B in its topology (see for example [3]).

2.1. **Real locally C*-algebras.** Let us now recall some notions and results about real locally C^*-algebras.

Definition 3. *A topological *-algebra A over $\mathbb{R}$, such that*

$$A \cap iA = \{\mathbf{0}\},$$

is called a ***hermitian real topological *-algebra*** *if*

$$\sigma(a) \subset \mathbb{R},$$

for each

$$a \in A_{sa},$$

i.e.

$$a = a^*,$$

where $\sigma(a)$ is a ***spectrum*** *of A, computed in its* ***complexification***

$$B = A \dot{+} iA.$$

Definition 4. *A topological *-algebra A over $\mathbb{R}$ is called a* ***real locally C^*-algebra*** *if there exists a projective family of real C^*-algebras*

$$\{A_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\},$$

so that

$$A \cong \varprojlim A_\alpha,$$

i.e. A is real topologically *-isomorphic to a projective limit of a projective family of real C^*-algebras, i.e. there exits an Arens-Michael decomposition of A composed entirely of real C^*-algebras.

Theorem 2 (Katz and Friedman [6]). *A topological *-algebra A over $\mathbb{R}$ is a real locally C^*-algebra iff A is a complete real hermitian Hausdorff topological *-algebra in which topology is generated by a saturated separating family of C^*-seminorms.*

Example 5. *Every real C^*-algebra is a locally C^*-algebra.*

Example 6. *A closed real *-subalgebra of a real locally C^*-algebra is a real locally C^*-algebra.*

Example 7. *The product $\prod\limits_{\alpha\in\Lambda} A_\alpha$ of real C^*-algebras A_α, with the product topology, is a real locally C^*-algebra.*

Example 8. *Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). Then the algebra $C(X)$ of all continuous, not necessarily bounded real-valued functions on X, with the topology of uniform convergence on compact subsets, is a real locally C^*-algebra.*

Katz and Friedman have presented in [6] that for each real locally C^*-algebra A, its complexification

$$B = A \dotplus iA,$$

can be endowed with topology turning it to a locally C^*-algebra. More specifically, the following theorem is valid:

Theorem 3 (Katz and Friedman [10]). *Let A be a real locally C^*-algebra, and $N(A)$ be the saturated separating family of continuous C^*-seminorms $\|.\|_\alpha$, $\alpha \in \Lambda$, generating the topology of A. Then there exists a saturated separating family $N(B)$ of continuous C^*-seminorms $\widetilde{\|.\|}_\alpha$, $\alpha \in \Lambda$, on its complexification*

$$B = A \dotplus iA,$$

such that $N(B)$ generates a topology an B turning it into a locally C^*-algebra, and such that $\widetilde{\|.\|}_\alpha$ extends $\|.\|_\alpha$ for each $\alpha \in \Lambda$.

Remark 1. *A detailed proof of the aforementioned result, which is using* [7] *and* [8]*, will appear in a separate publication.*

Let A be a locally real C^*-algebra. Then an element $a \in A$ is called **bounded**, if

$$\|a\|_\infty = \{\sup \|a\|_\alpha, \alpha \in \Lambda : \|.\|_\alpha \in N(A)\} < \infty.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is known that for each real locally C^*-algebra A, its set $b(A)$ of bounded elements of A is a real locally C^*-subalgebra, which is a real C^*-algebra in the norm $\|.\|_\infty$, such that it is dense in A in its topology, and

$$b(B) = b(A) \dotplus ib(A),$$

(cf. [6]), where $b(B)$ is the set of all bounded elements of

$$B = A \dotplus iA.$$

3. The uniqueness of involuton in real locally C*-algebras

One of proofs of the uniqueness of involution for real locally C^*-algebra can be obtained by following the footsteps of the proof of Katz from [9] with the usage of Theorem 2. In the current notes, however, we will show how to prove this result using the technique of complexification with Theorem 3, and then applying the result from [9].

Theorem 4. *The unary operation of involution in any real locally C^*-algebra is unique, i.e., if $(A,^*, \|.\|_\alpha, \alpha \in \Lambda)$ and $(A,^\#, \|.\|_\alpha, \alpha \in \Lambda)$ are two real locally C^*-algebras, means that each seminorm $\|.\|_\alpha, \alpha \in \Lambda$, satisfies the C^*-property for both operations, " $*$ " and "#", then*

$$^* = \#$$

on A.

Proof. Let us assume on the contrary, that

$$^* \neq \#,$$

on A, i.e. there exists $x \in A$, such that

$$x^* \neq x^\#.$$

Let now A be a real locally C^*-algebra, and let $N(A)$ be the saturated separating family of continuous C^*-seminorms, and each seminorm $\|.\|_\alpha, \alpha \in \Lambda$, satisfies the C^*-property for both operations, " $*$ " and "#". Let

$$B = A \dotplus iA.$$

From Theorem 3 it follows that the family $N(A)$ can be extended to saturated separating family $N(B)$ of continuous C^*-seminorms $\widetilde{\|.\|}_\alpha$, $\alpha \in \Lambda$, on B, such that $(B,^*, \widetilde{\|.\|}_\alpha, \alpha \in \Lambda)$ and $(B,^\#, \widetilde{\|.\|}_\alpha, \alpha \in \Lambda)$ are two locally C^*-algebras, means that each seminorm $\widetilde{\|.\|}_\alpha$, $\alpha \in \Lambda$, satisfies the C^*-property for both operations, " $*$ " and "#", and $(A,^*, \|.\|_\alpha, \alpha \in \Lambda)$ is a real locally C^*-subalgebras of a locally C^*-algebra $(B,^*, \widetilde{\|.\|}_\alpha, \alpha \in \Lambda)$, as well as $(A,^\#, \|.\|_\alpha, \alpha \in \Lambda)$ is a real locally C^*-subalgebra of a locally C^*-algebra $(B,^\#, \widetilde{\|.\|}_\alpha, \alpha \in \Lambda)$. Because without the loss of generality we can assume that

$$A \subset B,$$

and the unary operations " $*$ " and "#" on B extend the unary operations with the same name on A, it follows that there exists an element $x \in B$, such that

$$x^* \neq x^\#,$$

which contradicts the main result of [9].

The found contradiction proves the theorem. □

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