

# A Critical Parabolic Sobolev Embedding via Littlewood-Paley Decomposition

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## Abstract

In this paper, we show a parabolic version of the Ogawa type inequality in Sobolev spaces. Our inequality provides an estimate of the  $L^\infty$  norm of a function in terms of its parabolic  $BMO$  norm, with the aid of the square root of the logarithmic dependency of a higher order Sobolev norm. The proof is mainly based on the Littlewood-Paley decomposition and a characterization of parabolic  $BMO$  spaces.

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## 1 Introduction and main results

In order to study the long-time existence of a certain class of singular parabolic problems, Ibrahim and Monneau [9] made use of a parabolic logarithmic Sobolev inequality. They proved that for  $f \in W_2^{2m,m}(\mathbb{R}^{n+1})$ ,  $m, n \in \mathbb{N}^*$  and  $2m > \frac{n+2}{2}$ , the following estimate takes place (with  $\log^+ x = \max(\log x, 0)$ ):

$$\|f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C(1 + \|f\|_{BMO^a(\mathbb{R}^{n+1})}(1 + \log^+ \|f\|_{W_2^{2m,m}(\mathbb{R}^{n+1})})), \quad (1.1.1)$$

for some constant  $C = C(m, n) > 0$ . Here  $BMO^a$  stands for the anisotropic Bounded Mean Oscillation space with the parabolic anisotropy  $a = (1, \dots, 1, 2) \in \mathbb{R}^{n+1}$  (see Definition 2.1), while  $W_2^{2m,m}$  stands for the parabolic Sobolev space (see Definition 2.2). The above estimate, after also being proved on a bounded domain

$$\Omega_T = (0, 1)^n \times (0, T) \subseteq \mathbb{R}^{n+1}, \tag{1.1.2}$$

was successfully applied in order to obtain some *a priori* bounds on the gradient of the solution of particular parabolic equations leading eventually to the long-time existence (see [9, Proposition 3.7] or [8, Theorem 1.3]). The bounded version of (1.1.1) (see [9, Theorem 1.2]) reads: if  $f \in W_2^{2m,m}(\Omega_T)$  with  $2m > \frac{n+2}{2}$ , then:

$$\|f\|_{L^\infty(\Omega_T)} \leq C(1 + \|f\|_{\overline{BMO}^a(\Omega_T)}(1 + \log^+ \|f\|_{W_2^{2m,m}(\Omega_T)})), \tag{1.1.3}$$

where  $C = C(m, n, T) > 0$  is a positive constant, and

$$\|f\|_{\overline{BMO}^a(\Omega_T)} = \|f\|_{BMO^a(\Omega_T)} + \|f\|_{L^1(\Omega_T)}. \tag{1.1.4}$$

Indeed, the fact that inequality (1.1.1) does not hold on  $\Omega_T$  with a positive constant  $C^* = C^*(m, n, T)$  can be easily understood by applying this inequality to the function  $f = (C^* + \epsilon) \in W_2^{2m,m}(\Omega_T)$  with  $\epsilon > 0$ . In this case  $\|f\|_{L^\infty(\Omega_T)} = C^* + \epsilon$ ,  $\|f\|_{BMO^a(\Omega_T)} = 0$ , and hence a contradiction. However, working on  $\mathbb{R}^{n+1}$ , the same function  $f$  could not be used since  $f \notin W_2^{2m,m}(\mathbb{R}^{n+1})$ . Let us indicate that both inequalities (1.1.1) and (1.1.3) still hold for vector-valued functions  $f = (f_1, \dots, f_n, f_{n+1}) \in (W_2^{2m,m}(\mathbb{R}^{n+1}))^{n+1}$  with  $2m > \frac{n+2}{2}$  and the natural change in norm.

The elliptic version of (1.1.1) was showed by Kozono and Taniuchi in [12]. Indeed, they have showed that for  $f \in W_p^s(\mathbb{R}^n)$ ,  $1 < p < \infty$ , the following estimate holds:

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + \|f\|_{BMO(\mathbb{R}^n)}(1 + \log^+ \|f\|_{W_p^s(\mathbb{R}^n)})), \quad sp > n, \tag{1.1.5}$$

for some  $C = C(n, p, s) > 0$ . Here  $BMO$  is the usual elliptic/isotropic bounded mean oscillation space (defined via Euclidean balls). The main advantage of (1.1.5) is that it was successfully applied in order to extend the blow-up criterion of solutions to the Euler equations originally given by Beale, Kato and Majda in [1]. This blow-up criterion was then refined by Kozono, Ogawa and Taniuchi [11], and by Ogawa [13], showing weaker regularity criterion.

The proof of inequality (1.1.1) is based on the analysis in anisotropic Lizorkin-Triebel, Besov, Sobolev and  $BMO^a$  spaces. This is made via Littlewood-Paley decomposition and various Sobolev embeddings. In fact, some of the technical arguments were inspired by Ogawa [13] in his proof of the sharp version of (1.1.5) that reads: if  $g \in L^2(\mathbb{R}^n)$  and  $f := \nabla g \in W_q^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  for  $n < q$ , then there exists a constant  $C = C(q) > 0$  such that:

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C(q) \left( 1 + \|f\|_{BMO(\mathbb{R}^n)} \left( \log^+(\|f\|_{W_q^1(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}) \right)^{1/2} \right). \tag{1.1.6}$$

It is worth mentioning that the original type of the logarithmic Sobolev inequalities (1.1.5) and (1.1.6) was found in Brézis and Gallouët [3]. The Brézis-Gallouët-Wainger inequality states that the  $L^\infty$  norm of a function can be estimated by the  $W_p^{n/p}$  norm with the partial aid of the  $W_r^s$  norm with  $s > n/r$  and  $1 \leq r \leq \infty$ . Precisely,

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C \left( (1 + \log(1 + \|f\|_{W_r^s(\mathbb{R}^n)})) \right)^{\frac{p-1}{p}} \tag{1.1.7}$$

holds for all  $f \in W_p^{n/p}(\mathbb{R}^n) \cap W_r^s(\mathbb{R}^n)$  with the normalization  $\|f\|_{W_p^{n/p}(\mathbb{R}^n)} = 1$ . Originally, Brézis and Gallouët [3] obtained (1.1.7) for the case  $n = p = r = s = 2$ , where they applied their inequality in order to prove global existence of solutions to the nonlinear Schrödinger equation. Later on, Brézis and Wainger [4] obtained (1.1.7) for the general case, and remarked that the power  $\frac{p-1}{p}$  in (1.1.7) is optimal in the sense that one can not replace it by any smaller power. However, it seems that little is known about the sharp constant in (1.1.7).

Coming back to inequalities (1.1.1), (1.1.5) and (1.1.6), the natural question that arises is the following: why does the inequality (1.1.1) seems to be the parabolic extension of (1.1.5) although the proof is inspired (as mentioned above) from that of (1.1.6) given by Ogawa [13]? The answer to this question is partially contained in [9, Remark 2.14] where the authors pointed out that the well-known relation between elliptic/isotropic Lizorkin-Triebel and  $BMO$  spaces (see [13, Proposition 2.3]) will not be used in the proof of (1.1.1) even though it seems to be valid (without giving a proof) in the parabolic/anisotropic framework. The relation is the following:

$$\dot{F}_{\infty,2}^{0,a} \simeq BMO^a, \tag{1.1.8}$$

where  $\dot{F}_{\infty,2}^{0,a}$  is the homogeneous parabolic Lizorkin-Triebel space (see Definition 2.3).

In this paper, we show a parabolic version of the logarithmic Sobolev inequality (1.1.6) basically using the equivalence (1.1.8) that is shown to be true (see Lemma 3.1). This answers the question raised above. Our study takes place on the whole space  $\mathbb{R}^{n+1}$  and on the bounded domain  $\Omega_T$ . A comparison (in some special cases) of our inequality with (1.1.1) is also discussed.

Before stating our main results, we define some terminology. A generic element in  $\mathbb{R}^{n+1}$  will be denoted by  $z = (x, t) \in \mathbb{R}^{n+1}$  where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is the spatial variable, and  $t \in \mathbb{R}$  is the time variable. For a given function  $g$ , the notation  $\partial_i g$  stands for the partial derivative with respect to the spatial variable:  $\partial_i g = \partial_{x_i} g := \frac{\partial g}{\partial x_i}$ ,  $i = 1, \dots, n$ . In this case  $\partial_{n+1} g = \partial_t g := \frac{\partial g}{\partial t}$ . We also denote  $\partial_x^s g$ ,  $s \in \mathbb{N}$ , any derivative with respect to  $x$  of order  $s$ . Moreover, we denote the space-time gradient by  $\nabla g := (\partial_1 g, \dots, \partial_n g, \partial_{n+1} g)$ . Finally, we denote  $\|f\|_X := \max(\|f_1\|_X, \dots, \|f_n\|_X, \|f_{n+1}\|_X)$  for any vector-valued function  $f = (f_1, \dots, f_n, f_{n+1}) \in X^{n+1}$  where  $X$  is any Banach space. Throughout this paper and for the sake of simplicity, we will drop the superscript  $n + 1$  from  $X^{n+1}$ . Following the above notations, our first theorem reads:

**Theorem 1.1** (*Parabolic Ogawa inequality on  $\mathbb{R}^{n+1}$* ). *Let  $m, n \in \mathbb{N}^*$  with  $2m > \frac{n+2}{2}$ . Then there exists a constant  $C = C(m, n) > 0$  such that for any function  $g \in L^2(\mathbb{R}^{n+1})$  with  $f = (f_1, \dots, f_n, f_{n+1}) = \nabla g \in W_2^{2m, m}(\mathbb{R}^{n+1})$ , we have:*

$$\|f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C \left( 1 + \|f\|_{BMO^a(\mathbb{R}^{n+1})} \left( \log^+ (\|f\|_{W_2^{2m, m}(\mathbb{R}^{n+1})} + \|g\|_{L^\infty(\mathbb{R}^{n+1})}) \right)^{1/2} \right). \quad (1.1.9)$$

**Remark 1.2** *Inequalities (1.1.1) and (1.1.9) have the same order of the higher regular term. As a consequence, inequality (1.1.9) can also be applied in order to establish the long-time existence of solutions of the parabolic problems studied in [8, 9].*

Our next theorem concerns a similar type inequality of (1.1.9), but with functions  $g$  and  $f$  defined over  $\Omega_T$  (given by (1.1.2)). Before stating this result, we first remark that in the case of functions  $f = \nabla g$  defined on a bounded domain, we formally have (by Poincaré inequality):

$$\|g\|_{L^\infty} \leq C \|f\|_{L^\infty},$$

where  $C > 0$  is a constant depending on the measure of the domain. Moreover, since

$$\|f\|_{L^\infty} \leq C_1 \|f\|_{C^{\gamma, \gamma/2}} \leq C_2 \|f\|_{W_2^{2m, m}} \quad \text{with } C_1, C_2 > 0,$$

the above two estimates imply that the term  $\|g\|_{L^\infty}$  should be dropped from inequality (1.1.9) when dealing with functions defined over bounded domains. Indeed, we have:

**Theorem 1.3** (*Parabolic Ogawa inequality on a bounded domain*). *Let  $f \in W_2^{2m,m}(\Omega_T)$  with  $2m > \frac{n+2}{2}$ . Then there exists a constant  $C = C(m, n, T) > 0$  such that:*

$$\|f\|_{L^\infty(\Omega_T)} \leq C \left( 1 + \|f\|_{\overline{BMO}^a(\Omega_T)} \left( \log^+ \|f\|_{W_2^{2m,m}(\Omega_T)} \right)^{1/2} \right), \quad (1.1.10)$$

where the norm  $\|\cdot\|_{\overline{BMO}^a(\Omega_T)}$  is given by (1.1.4).

**Remark 1.4** *Inequality (1.1.10) is sharper than (1.1.3) by the simple observation that  $x^{1/2} \leq 1 + x$  for  $x \geq 0$ . In other words, inequality (1.1.10) implies (1.1.3) with the same positive constant  $C = C(m, n)$ .*

In the same spirit of Remark 1.4, our last theorem gives a comparison between inequality (1.1.1) and (1.1.9) for a certain class of functions  $g$ , and for particular space dimensions.

**Theorem 1.5** (*Comparison between parabolic logarithmic inequalities*). *Let  $n = 1, 2, 3$  and  $m \in \mathbb{N}^*$  satisfying  $2m > \frac{n+2}{2}$ . There exists a constant  $C = C(m, n) > 0$  such that for the class of functions  $g \in L^2(\mathbb{R}^{n+1})$  with  $\|g\|_{L^2(\mathbb{R}^{n+1})} \leq 1$ , and  $f = \nabla g \in W_2^{2m,m}(\mathbb{R}^{n+1})$ , we have:*

$$\left( \log^+ (\|f\|_{W_2^{2m,m}(\mathbb{R}^{n+1})} + \|g\|_{L^\infty(\mathbb{R}^{n+1})}) \right)^{1/2} \leq C (1 + \log^+ \|f\|_{W_2^{2m,m}(\mathbb{R}^{n+1})}), \quad (1.1.11)$$

and hence inequality (1.1.9) implies (1.1.1) for possibly a different positive constant  $C$ .

## 1.1 Organization of the paper

This paper is organized as follows. In Section 2, we present some definitions and the main tools used in our analysis. This includes parabolic Littlewood-Paley decomposition and various Sobolev embeddings. Section 3 is devoted to the proof of Theorem 1.1 (estimate on the entire space  $\mathbb{R}^{n+1}$ ) using mainly the equivalence (1.1.8) that we also show in Lemma 3.1. In Section 4, we give the proof of Theorem 1.3 (estimate on the bounded domain  $\Omega_T$ ). Finally, in Section 5, we give the proof of Theorem 1.5.

## 2 Preliminaries and basic tools

In this section, we define the fundamental function spaces used in this paper. We also recall some important embeddings.

### 2.1 Parabolic $BMO^a$ and Sobolev spaces

Each coordinate  $x_i, i = 1, \dots, n$  is given the weight 1, while the time coordinate  $t$  is given the weight 2. The vector  $a = (a_1, \dots, a_n, a_{n+1}) = (1, \dots, 1, 2) \in \mathbb{R}^{n+1}$  is called the  $(n + 1)$ -dimensional parabolic anisotropy. For this given  $a$ , the action of  $\mu \in [0, \infty)$  on  $z = (x, t)$  is given by  $\mu^a z = (\mu x_1, \dots, \mu x_n, \mu^2 t)$ . For  $\mu > 0$  and  $s \in \mathbb{R}$  we set  $\mu^{sa} z = (\mu^s)^a z$ . In particular,  $\mu^{-a} z = (\mu^{-1})^a z$  and  $2^{-ja} z = (2^{-j})^a z, j \in \mathbb{Z}$ . For  $z \in \mathbb{R}^{n+1}, z \neq 0$ , let  $|z|_a$  be the unique positive number  $\mu$  such that:

$$\frac{x_1^2}{\mu^2} + \dots + \frac{x_n^2}{\mu^2} + \frac{t^2}{\mu^4} = 1$$

and let  $|z|_a = 0$  for  $z = 0$ . The map  $|\cdot|_a$  is called the parabolic distance function which is  $C^\infty$ . In the case where  $a = (1, \dots, 1) \in \mathbb{R}^{n+1}$ , we get the usual Euclidean distance  $\|z\| = (x_1^2 + \dots + x_n^2 + t^2)^{1/2}$ . Denoting  $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ , any open subset of  $\mathbb{R}^{n+1}$ , we are ready to give the definition of the first two parabolic spaces used in our analysis.

**Definition 2.1** (Parabolic bounded mean oscillation spaces). A function  $f \in L^1_{loc}(\mathcal{O})$  (defined up to an additive constant) is said to be of parabolic bounded mean oscillation,  $f \in BMO^a(\mathcal{O})$ , if we have:

$$\|f\|_{BMO^a(\mathcal{O})} = \sup_{\mathcal{Q} \subseteq \mathcal{O}} \inf_{c \in \mathbb{R}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |f - c| \right) < +\infty, \tag{2.2.1}$$

where  $\mathcal{Q}$  denotes (for  $z_0 \in \mathcal{O}$  and  $r > 0$ ) an arbitrary parabolic cube:

$$\mathcal{Q} = \mathcal{Q}_r(z_0) = \{z \in \mathbb{R}^{n+1}; |z - z_0|_a < r\}.$$

**Definition 2.2** (Parabolic Sobolev spaces). Let  $m \in \mathbb{N}$ . We define the parabolic Sobolev space  $W_2^{2m,m}(\mathcal{O})$  as follows:

$$W_2^{2m,m}(\mathcal{O}) = \{f \in L^2(\mathcal{O}); \partial_t^r \partial_x^s f \in L^2(\mathcal{O}), \forall r, s \in \mathbb{N} \text{ such that } 2r + s \leq 2m\},$$

with  $\|f\|_{W_2^{2m,m}(\mathcal{O})} = \sum_{j=0}^{2m} \sum_{2r+s=j} \|\partial_t^r \partial_x^s f\|_{L^2(\mathcal{O})}$ .

## 2.2 Parabolic Lizorkin-Triebel and Besov spaces

Along with the above parabolic distance  $|\cdot|_a$ , the Littlewood-Paley decomposition is now recalled. Let  $\theta \in C_0^\infty(\mathbb{R}^{n+1})$  be any cut-off function satisfying:

$$\begin{aligned} \theta(z) &= 1 \quad \text{if } |z|_a \leq 1 \\ \theta(z) &= 0 \quad \text{if } |z|_a \geq 2. \end{aligned} \tag{2.2.2}$$

Let  $\psi(z) = \theta(z) - \theta(2^a z)$ . We now construct a smooth (compactly supported) parabolic dyadic partition of unity  $(\psi_j)_{j \in \mathbb{Z}}$  by letting

$$\psi_j(z) = \psi(2^{-ja} z), \quad j \in \mathbb{Z}, \tag{2.2.3}$$

satisfying  $\sum_{j \in \mathbb{Z}} \psi_j(z) = 1$  for  $z \neq 0$ . Define  $\varphi_j, j \in \mathbb{Z}$ , as the inverse Fourier transform of  $\psi_j$ , i.e.  $\hat{\varphi}_j = \psi_j$  where we let

$$\varphi := \varphi_0. \tag{2.2.4}$$

It is worth noticing that  $\varphi_j$  satisfies:

$$\varphi_j(z) = 2^{(n+2)j} \varphi(2^{ja} z), \quad j \in \mathbb{Z} \quad \text{and} \quad z \in \mathbb{R}^{n+1}. \tag{2.2.5}$$

The above Littlewood-Paley decomposition asserts that any tempered distribution  $f \in \mathcal{S}'(\mathbb{R}^{n+1})$  can be decomposed as:

$$f = \sum_{j \in \mathbb{Z}} \varphi_j * f \quad \text{with the convergence in } \mathcal{S}'/\mathcal{P} \text{ (modulo polynomials).}$$

Here  $\mathcal{S}(\mathbb{R}^{n+1})$  is the usual Schwartz class of rapidly decreasing functions and  $\mathcal{S}'(\mathbb{R}^{n+1})$  is its corresponding dual, represents the space of tempered distributions. We now define parabolic Lizorkin-Triebel spaces.

**Definition 2.3** (*Parabolic homogeneous Lizorkin-Triebel spaces*). Given a smoothness parameter  $s \in \mathbb{R}$ , an integrability exponent  $1 \leq p < \infty$ , and a summability exponent  $1 \leq q \leq \infty$  ( $1 \leq q < \infty$  if  $p = \infty$ ). Let  $\varphi_j$  be given by (2.2.5), we define the parabolic homogeneous Lizorkin-Triebel space  $\dot{F}_{p,q}^{s,a}$  as the space of all functions  $f \in \mathcal{S}'(\mathbb{R}^{n+1})$  with finite quasi-norms

$$\|f\|_{\dot{F}_{p,q}^{s,a}(\mathbb{R}^{n+1})} = \left\| \left( \sum_{j \in \mathbb{Z}} 2^{sqj} |\varphi_j * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n+1})} < \infty,$$

and the natural modification for  $q = \infty$ .

As a convention, for  $s \in \mathbb{R}$ , and  $1 \leq q < \infty$ , we denote

$$\|f_+\|_{\dot{F}_{\infty,q}^{s,a}(\mathbb{R}^{n+1})} = \left\| \left( \sum_{j \geq 1} 2^{sqj} |\varphi_j * f|^q \right)^{1/q} \right\|_{L^\infty(\mathbb{R}^{n+1})} \tag{2.2.6}$$

and

$$\|f_-\|_{\dot{F}_{\infty,q}^{s,a}(\mathbb{R}^{n+1})} = \left\| \left( \sum_{j \leq -1} 2^{sqj} |\varphi_j * f|^q \right)^{1/q} \right\|_{L^\infty(\mathbb{R}^{n+1})}. \tag{2.2.7}$$

The space  $\dot{F}_{p,2}^{0,a}$  can be identified with the parabolic Hardy space  $H^{p,a}(\mathbb{R}^{n+1})$ ,  $1 \leq p < \infty$ , having the following square function characterization stated informally as:

$$H^{p,a}(\mathbb{R}^{n+1}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^{n+1}); \left( \sum_{j \in \mathbb{Z}} |\varphi_j * f|^2 \right)^{1/2} \in L^p \right\}. \tag{2.2.8}$$

This identification between the above two spaces is the following:

**Theorem 2.4** (*Identification between  $H^{p,a}$  and  $\dot{F}_{p,2}^{0,a}$* ). (See Bownik [2].) For all  $1 \leq p < \infty$ , we have  $\dot{F}_{p,2}^{0,a}(\mathbb{R}^{n+1}) \simeq H^{p,a}(\mathbb{R}^{n+1})$ .

Another useful space throughout our analysis is the parabolic inhomogeneous Besov space. The main difference in defining this space is the choice of the parabolic dyadic partition of unity that is now altered. Indeed, we take  $(\psi_j)_{j \geq 0}$  satisfying:

$$\begin{aligned} \psi_j &:= \psi_j \text{ defined by (2.2.3) if } j \geq 1 \\ \psi_j &:= \theta \text{ defined by (2.2.2) if } j = 0. \end{aligned} \tag{2.2.9}$$

Again, it is clear that  $\sum_{j \geq 0} \psi_j(z) = 1$ , but now for all  $z \in \mathbb{R}^{n+1}$ , and in exactly the same way as above, we can rewrite the Littlewood-Paley decomposition with

$$\hat{\varphi}_j = \psi_j, \quad j \geq 0, \quad \psi_j \text{ is given by (2.2.9)}. \tag{2.2.10}$$

We then arrive to the following definition:

**Definition 2.5** (*Parabolic inhomogeneous Besov spaces*). Given a smoothness parameter  $s \in \mathbb{R}$ , an integrability exponent  $1 \leq p \leq \infty$ , and a summability exponent  $1 \leq q \leq \infty$ , we define the parabolic inhomogeneous Besov space  $B_{p,q}^{s,a}$  as the space of all functions  $u \in \mathcal{S}'(\mathbb{R}^{n+1})$  with finite quasi-norms

$$\|u\|_{B_{p,q}^{s,a}} = \left( \sum_{j \geq 0} 2^{sqj} \|\varphi_j * u\|_{L^p(\mathbb{R}^{n+1})}^q \right)^{1/q} < \infty, \quad \varphi_j \text{ is given by (2.2.10)}$$

and the natural modification for  $q = \infty$ , i.e.

$$\|u\|_{B_{p,\infty}^{s,a}} = \sup_{j \geq 0} 2^{sj} \|\varphi_j * u\|_{L^p(\mathbb{R}^{n+1})}, \quad \varphi_j \text{ is given by (2.2.10)}. \quad (2.2.11)$$

For a detailed study of anisotropic Lizorkin-Triebel and Besov spaces, we refer the reader to Triebel [16].

### 2.3 Embeddings of parabolic Besov and Sobolev spaces

We present two embedding results from Johnsen and Sickel [10], and Stöckert [14].

**Theorem 2.6** (*Embeddings of Besov spaces*). (See Johnsen and Sickel [10].) Let  $s, t \in \mathbb{R}$ ,  $s > t$ , and  $1 \leq p, r \leq \infty$  satisfy:  $s - \frac{n+2}{p} = t - \frac{n+2}{r}$ . Then for any  $1 \leq q \leq \infty$  we have the following continuous embedding:

$$B_{p,q}^{s,a}(\mathbb{R}^{n+1}) \hookrightarrow B_{r,q}^{t,a}(\mathbb{R}^{n+1}). \quad (2.2.12)$$

**Proposition 2.7** (*Sobolev embeddings in Besov spaces*). (See Stöckert [14].) Let  $m \in \mathbb{N}$ , then we have:

$$W_2^{2m,m}(\mathbb{R}^{n+1}) \hookrightarrow B_{2,\infty}^{2m,a}(\mathbb{R}^{n+1}). \quad (2.2.13)$$

## 3 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We start by showing the equivalence (1.1.8) whose isotropic version can be found in Triebel [15], and Frazier and Jawerth [6].

**Lemma 3.1** (*Equivalence between  $\dot{F}_{\infty,2}^{0,a}$  and  $BMO^a$* ). We have  $\dot{F}_{\infty,2}^{0,a}(\mathbb{R}^{n+1}) \simeq BMO^a(\mathbb{R}^{n+1})$ . Precisely, there exists a constant  $C > 0$  such that:

$$C^{-1} \|f\|_{\dot{F}_{\infty,2}^{0,a}} \leq \|f\|_{BMO^a} \leq C \|f\|_{\dot{F}_{\infty,2}^{0,a}}. \quad (3.3.1)$$

**Proof.** See [7, Lemma 3.2, page 6]. □

A basic estimate is now shown in the following lemma.

**Lemma 3.2** (*Logarithmic estimate in parabolic Lizorkin-Triebel spaces*). Let  $\gamma > 0$  be a positive real number. Then, for  $f \in \dot{F}_{\infty,1}^{0,a}$  with  $\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}(\mathbb{R}^{n+1})}$  and

$\|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}(\mathbb{R}^{n+1})}$  are both finite, there exists a constant  $C = C(n, \gamma) > 0$  such that:

$$\|f\|_{\dot{F}_{\infty,1}^{0,a}} \leq C \left( 1 + \|f\|_{\dot{F}_{\infty,2}^{0,a}} \left( \log^+ (\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}) \right)^{1/2} \right). \quad (3.3.2)$$

**Proof.** We first indicate that the constant  $C = C(n, \gamma) > 0$  may vary from line to line in the proof which is divided into two steps.

**Step 1** (First estimate on  $\|f\|_{\dot{F}_{\infty,1}^{0,a}}$ ). Let  $N \in \mathbb{N}$ . We compute

$$\begin{aligned} \|f\|_{\dot{F}_{\infty,1}^{0,a}} &\leq \left\| \sum_{j < -N} 2^{\gamma j} 2^{-\gamma j} |\varphi_j * f| \right\|_{L^\infty} + \left\| \sum_{|j| \leq N} |\varphi_j * f| \right\|_{L^\infty} + \left\| \sum_{j > N} 2^{-\gamma j} 2^{\gamma j} |\varphi_j * f| \right\|_{L^\infty} \\ &\leq C_\gamma 2^{-\gamma N} \left\| \left( \sum_{j < -N} 2^{-2\gamma j} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} + (2N + 1)^{1/2} \left\| \left( \sum_{|j| \leq N} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} \\ &\quad + C_\gamma 2^{-\gamma N} \left\| \left( \sum_{j > N} 2^{2\gamma j} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} \\ &\leq C_\gamma 2^{-\gamma N} \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}} + C(2N + 1)^{1/2} \|f\|_{\dot{F}_{\infty,2}^{0,a}} + C_\gamma 2^{-\gamma N} \|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}}, \end{aligned}$$

with  $C_\gamma = \left(\frac{1}{2^{2\gamma}-1}\right)^{1/2}$ . As a conclusion we may write

$$\|f\|_{\dot{F}_{\infty,1}^{0,a}} \leq C \left( (2N + 1)^{1/2} \|f\|_{\dot{F}_{\infty,2}^{0,a}} + 2^{-\gamma N} (\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}) \right). \quad (3.3.3)$$

**Step 2** (Optimization in  $N$ ). We optimize (3.3.3) in  $N$  by setting:

$$N = 1 \quad \text{if} \quad \|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}} \leq 2^\gamma \|f\|_{\dot{F}_{\infty,2}^{0,a}}.$$

Then it is easy to check (using (3.3.3)) that

$$\|f\|_{\dot{F}_{\infty,1}^{0,a}} \leq C \|f\|_{\dot{F}_{\infty,2}^{0,a}} \left( 1 + \left( \log^+ \frac{\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}}{\|f\|_{\dot{F}_{\infty,2}^{0,a}}} \right)^{1/2} \right). \quad (3.3.4)$$

In the case where  $\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}} > 2^\gamma \|f\|_{\dot{F}_{\infty,2}^{0,a}}$ , we take  $1 \leq \beta < 2^\gamma$  such that

$$N = N(\beta) = \log_{2^\gamma}^+ \left( \beta \frac{\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}}{\|f\|_{\dot{F}_{\infty,2}^{0,a}}} \right) - \frac{1}{2} \in \mathbb{N}.$$

In fact this is valid since the function  $N(\beta)$  varies continuously from  $N(1)$  to  $N(2^\gamma) = 1 + N(1)$  on the interval  $[1, 2^\gamma]$ . Using (3.3.3) with the above choice

of  $N$ , we obtain:

$$\begin{aligned} \|f\|_{\dot{F}_{\infty,1}^{0,a}} &\leq C \left[ 2^{1/2} \left( \log_{2^\gamma}^+ \left( \beta \frac{\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}}{\|f\|_{\dot{F}_{\infty,2}^{0,a}}} \right) \right)^{1/2} \|f\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \frac{2^{\gamma/2}}{\beta} \|f\|_{\dot{F}_{\infty,2}^{\gamma,a}} \right] \\ &\leq C \left[ \frac{2}{(\gamma \log 2)^{1/2}} \left( \log^+ \left( \frac{\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}}{\|f\|_{\dot{F}_{\infty,2}^{0,a}}} \right) \right)^{1/2} \|f\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \frac{2^{\gamma/2}}{\beta} \|f\|_{\dot{F}_{\infty,2}^{\gamma,a}} \right], \end{aligned}$$

where for the second line we have used the fact that

$$\log^+ \beta < \log^+ \frac{\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} + \|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}}{\|f\|_{\dot{F}_{\infty,2}^{0,a}}}.$$

The above computations again imply (3.3.4). By using the inequality:

$$x \left( \log \left( e + \frac{y}{x} \right) \right)^{1/2} \leq C(1 + x(\log(e + y))^{1/2}) \quad \text{for } 0 < x \leq 1,$$

$$x \left( \log \left( e + \frac{y}{x} \right) \right)^{1/2} \leq Cx(\log(e + y))^{1/2} \quad \text{for } x > 1, \tag{3.3.5}$$

in (3.3.4), we directly arrive to our result. □

We now present the proof of our first main result.

**Proof of Theorem 1.1.** First let us mention that the constant  $C = C(m, n) > 0$  appearing in the following proof may vary from line to line. We will show inequality (1.1.9) in the scalar-valued version, i.e. by considering  $f = f_i = \partial_i g$  for some fixed  $i = 1, \dots, n + 1$ . The vector-valued version can then be easily deduced. The proof requires estimating all the terms of inequality (3.3.2). We start with the obvious estimate (see (3.3.1)):

$$\|f\|_{\dot{F}_{\infty,2}^{0,a}} \leq C\|f\|_{BMO^a}. \tag{3.3.6}$$

The remaining terms will be estimated in the following three steps.

**Step 1** (An upper bound on  $\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}}$ ). Set  $\eta = 2m - \frac{n+2}{2} > 0$ . Choose  $\gamma$  such that:

$$0 < \gamma < \eta.$$

We compute (see (2.2.6)):

$$\begin{aligned} \|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} &= \left\| \left( \sum_{j \geq 1} 2^{2\gamma j} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} \\ &\leq C \sup_{j \geq 1} 2^{\eta j} \|\varphi_j * f\|_{L^\infty} \end{aligned} \tag{3.3.7}$$

with  $C = \left( \sum_{j \geq 1} 2^{2(\gamma-\eta)j} \right)^{1/2} < +\infty$ . Note that the sequence of functions  $(\varphi_j)_{j \geq 1}$  given in (3.3.7) can be identified with those given in (2.2.11). Hence we may write

$$\sup_{j \geq 1} 2^{\eta j} \|\varphi_j * f\|_{L^\infty} \leq \sup_{j \geq 0} 2^{\eta j} \|\varphi_j * f\|_{L^\infty}, \quad \varphi_j \text{ is given by (2.2.10),}$$

and then (using (3.3.7)) we obtain:

$$\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} \leq C \|f\|_{B_{\infty,\infty}^{\eta,a}}. \tag{3.3.8}$$

Using (2.2.12) with  $s = 2m, p = 2, q = \infty, t = \eta$  and  $r = \infty$ , we deduce that:

$$B_{2,\infty}^{2m,a} \hookrightarrow B_{\infty,\infty}^{\eta,a}.$$

Therefore, by (2.2.13), we get

$$W_2^{2m,m} \hookrightarrow B_{2,\infty}^{2m,a} \hookrightarrow B_{\infty,\infty}^{\eta,a}$$

which, together with (3.3.8), give:

$$\|f_+\|_{\dot{F}_{\infty,2}^{\gamma,a}} \leq C \|f\|_{W_2^{2m,m}}. \tag{3.3.9}$$

**Step 2** (An upper bound on  $\|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}}$ ). In this step, we will use the fact that  $\partial_i g = f_i$  (for which we keep denoting it by  $f$ , i.e.  $f = f_i$ ) for some  $i = 1, \dots, n + 1$ , with  $g \in L^\infty(\mathbb{R}^{n+1})$ . For  $z \in \mathbb{R}^{n+1}$ , define

$$\Phi(z) = (\partial_i \varphi)(z), \quad \varphi \text{ is given by (2.2.4),} \tag{3.3.10}$$

and

$$\Phi_j(z) = 2^{(n+2)j} \Phi(2^j z) \quad \text{for all } j \leq -1. \tag{3.3.11}$$

Using (2.2.5) we obtain:

$$(\partial_i \varphi_j)(z) = 2^j \Phi_j(z) \quad \text{if } i = 1, \dots, n$$

$$(\partial_i \varphi_j)(z) = 2^{2j} \Phi_j(z) \quad \text{if } i = n + 1. \tag{3.3.12}$$

We now compute (see (2.2.7), (3.3.11) and (3.3.12)):

$$\|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}} = \left\| \left( \sum_{j \leq -1} 2^{-2\gamma j} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} \tag{3.3.13}$$

$$\leq C \sup_{j \leq -1} \|\Phi_j * g\|_{L^\infty}, \tag{3.3.14}$$

where the constant  $C$  is given by:

$$C^2 = \sum_{j \leq -1} 2^{2j(1-\gamma)} \quad \text{if } i = 1, \dots, n$$

$$C^2 = \sum_{j \leq -1} 2^{2j(2-\gamma)} \quad \text{if } i = n + 1,$$

which is finite  $0 < C < +\infty$  under the choice

$$0 < \gamma < 1.$$

In order to terminate the proof, it suffices to show that

$$\|\Phi_j * g\|_{L^\infty} \leq C \|g\|_{L^\infty},$$

which can be deduced, by translation and dilation invariance, from the following estimate:

$$|(\Phi * g)(0)| \leq C \|g\|_{L^\infty}. \tag{3.3.15}$$

Indeed, define the positive radial decreasing function  $h(r) = h(\|z\|)$  as follows:

$$h(r) = \sup_{\|z\| \geq r} |\Phi(z)|.$$

From (3.3.10), we remark that the function  $\Phi$  is the inverse Fourier transform of a compactly supported function. Hence, we have:

$$h(0) = \|\Phi\|_{L^\infty} < +\infty, \tag{3.3.16}$$

and the asymptotic behavior

$$h(r) \leq \frac{C}{r^{n+2}} \quad \text{for all } r \geq 1. \tag{3.3.17}$$

We compute (taking  $S_r^n$  as the  $n$ -dimensional sphere of radius  $r$ ):

$$\begin{aligned}
 |(\Phi * g)(0)| &\leq \int_{\mathbb{R}^{n+1}} |\Phi(-z)||g(z)|dz \\
 &\leq \int_0^\infty \left( \int_{S_r^n} |\Phi(-z)||g(z)|d\sigma(z) \right) dr \\
 &\leq C \left( \int_0^\infty r^n h(r)dr \right) \|g\|_{L^\infty}.
 \end{aligned}
 \tag{3.3.18}$$

Using (3.3.16) and (3.3.17) we deduce that:

$$\begin{aligned}
 \int_0^\infty r^n h(r)dr &= \int_0^1 r^n h(r)dr + \int_1^\infty r^n h(r)dr \\
 &\leq C \left( \int_0^1 h(0)dr + \int_1^\infty \frac{r^n}{r^{n+2}}dr \right) \\
 &\leq C(\|\Phi\|_{L^\infty} + 1)
 \end{aligned}$$

which, together with (3.3.18), directly implies (3.3.15). As a conclusion, we obtain (see (3.3.13)):

$$\|f_-\|_{\dot{F}_{\infty,2}^{-\gamma,a}} \leq C\|g\|_{L^\infty}.
 \tag{3.3.19}$$

**Step 3** (A lower bound on  $\|f\|_{\dot{F}_{\infty,1}^{0,a}}$  and conclusion). *Remarking that*

$$\|f\|_{L^\infty} = \left\| \sum_{j \in \mathbb{Z}} \varphi_j * f \right\|_{L^\infty} \leq \|f\|_{\dot{F}_{\infty,1}^{0,a}}$$

when  $\hat{f}(0) = 0$ , the estimates (3.3.2), (3.3.6), (3.3.9) and (3.3.19) lead directly to the proof. □

### 4 Proof of Theorem 1.3

For the sake of simplicity, we only give the proof in the framework of one spatial dimensions  $x = x_1$ . The extension to the multi spatial dimensions can be easily deduced and will be made clear later in this section. Again, the constant  $C > 0$  that will appear in the following proof may vary from line to line but will only depend on  $m$  and  $T$

**Proof of Theorem 1.3.** We first remark that the function  $f$  can be extended by continuity to the boundary  $\partial\Omega_T$  of  $\Omega_T$ . Following the same notations of Ibrahim and Monneau [9], we take  $\tilde{f}$  as the extension of  $f$  over

$$\tilde{\Omega}_T = (-1, 2) \times (-T, 2T)$$

given by:

$$\tilde{f}(x, t) = \sum_{j=0}^{2m-1} c_j f(-\lambda_j x, t) \quad \text{for } -1 < x < 0, 0 \leq t \leq T \quad (4.4.1)$$

$$\tilde{f}(x, t) = \sum_{j=0}^{2m-1} c_j f(1 + \lambda_j(1 - x), t) \quad \text{for } 1 < x < 2, 0 \leq t \leq T,$$

with  $\lambda_j = \frac{1}{2^j}$  and

$$\tilde{f}(x, t) = \sum_{j=0}^{2m-1} c_j (-\lambda_j)^k = 1 \quad \text{for } k = 0, \dots, 2m - 1.$$

For the extension with respect to the time variable  $t$ , we use the same extension (4.4.1) summing up only to  $m - 1$ . The above extension (4.4.1) has been made in order to have (see for instance Evans [5])  $\tilde{f} \in W_2^{2m,m}(\tilde{\Omega}_T)$  and

$$\|\tilde{f}\|_{W_2^{2m,m}(\tilde{\Omega}_T)} \leq C \|f\|_{W_2^{2m,m}(\Omega_T)}. \quad (4.4.2)$$

Now let  $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$  be two subsets of  $\tilde{\Omega}_T$  defined by:

$$\mathcal{Z}_1 = \{(x, t); -1/4 < x < 5/4 \text{ and } -T/4 < t < 5T/4\}$$

and

$$\mathcal{Z}_2 = \{(x, t); -3/4 < x < 7/4 \text{ and } -3T/4 < t < 7T/4\}.$$

We take the cut-off function  $\Psi \in C_0^\infty(\mathbb{R}^2)$ ,  $0 \leq \Psi \leq 1$  satisfying:

$$\Psi(x, t) = 1 \quad \text{for } (x, t) \in \mathcal{Z}_1 \quad (4.4.3)$$

$$\Psi(x, t) = 0 \quad \text{for } (x, t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2.$$

From (4.4.2), we easily deduce that  $\Psi \tilde{f} \in W_2^{2m,m}(\mathbb{R}^2)$  and

$$\|\Psi \tilde{f}\|_{W_2^{2m,m}(\mathbb{R}^2)} \leq C \|f\|_{W_2^{2m,m}(\Omega_T)}. \quad (4.4.4)$$

Hence we can apply the scalar-valued version of inequality (1.1.9) (see Remark ??) with  $i = 1$ , i.e.  $\partial_1 g = f$ ; the new function (for which we give the same notation)  $f = \Psi \tilde{f} \in W_2^{2m,m}(\mathbb{R}^2)$  and  $g \in L^\infty(\mathbb{R}^2)$  given by

$$g(x, t) = \int_0^x \Psi(y, t) \tilde{f}(y, t) dy.$$

Since  $\Psi \tilde{f}$  is of compact support, and (again by the extension (4.4.1))  $\|\tilde{f}\|_{L^\infty(\tilde{\Omega}_T)} \leq C\|f\|_{W_2^{2m,m}(\Omega_T)}$ , we deduce that

$$\|g\|_{L^\infty(\mathbb{R}^2)} \leq C\|\tilde{f}\|_{L^\infty(\tilde{\Omega}_T)} \leq C\|f\|_{W_2^{2m,m}(\Omega_T)}. \tag{4.4.5}$$

Collecting the above arguments (namely (4.4.4) and (4.4.5)) together with the fact that (see Ibrahim and Monneau [9])

$$\|\Psi \tilde{f}\|_{BMO^a(\mathbb{R}^2)} \leq C\|f\|_{\overline{BMO}^a(\Omega_T)},$$

inequality (1.1.9) gives:

$$\begin{aligned} \|f\|_{L^\infty(\Omega_T)} \leq \|\Psi \tilde{f}\|_{L^\infty(\mathbb{R}^2)} &\leq C \left( 1 + \|\Psi \tilde{f}\|_{BMO^a(\mathbb{R}^2)} \left( \log^+(\|\Psi \tilde{f}\|_{W_2^{2m,m}(\mathbb{R}^2)} + \|g\|_{L^\infty(\mathbb{R}^2)}) \right)^{1/2} \right) \\ &\leq C \left( 1 + \|f\|_{\overline{BMO}^a(\Omega_T)} \left( \log^+ \|f\|_{W_2^{2m,m}(\Omega_T)} \right)^{1/2} \right). \end{aligned}$$

Notice that in the first line of the above inequalities we have used that  $\Psi = 1$  in  $\Omega_T$ . □

**Remark 4.1** *The inequality (1.1.9) used in the previous proof could have also been used for  $i = 2$ . In this case we simply take  $g(x, t) = \int_0^t \Psi(x, s)\tilde{f}(x, s)ds$ .*

**Remark 4.2** *In the case of multi spatial dimensions  $x_i, i = 1, \dots, n$ , we simultaneously apply the extension (4.4.1) to each spatial coordinate while fixing all the other coordinates including time  $t$ . However, the extension with respect to  $t$  is kept unchanged.*

## 5 Comparison between parabolic logarithmic inequalities

In this section we give the proof of Theorem 1.5. Throughout all this section, we only consider isotropic function spaces, i.e.  $a = (1, \dots, 1) \in \mathbb{R}^{n+1}$ . We only deal with the parabolic function space  $W_2^{2m,m}$ . As usual, the constant  $C = C(m, n) > 0$  may differ from line to line. First of all, we remark that estimate (1.1.11) turns out to be true (using the trivial identity  $x^{1/2} \leq 1 + x$ ) if  $A := \|g\|_{L^\infty} \leq C$  for  $\|f\|_{W_2^{2m,m}} \leq 1$ , or if  $B := \|g\|_{L^\infty}/\|f\|_{W_2^{2m,m}} \leq C$  for  $\|f\|_{W_2^{2m,m}} \geq 1$ . This will be proved in the forthcoming arguments. We start with the following lemmas:

**Lemma 5.1** *Let  $n = 1, 2, 3$ ,  $s = \frac{n+1}{2}$ , and  $m \in \mathbb{N}^*$  satisfying  $2m > \frac{n+2}{2}$ . For any  $g \in L^2(\mathbb{R}^{n+1})$  with  $f = \nabla g \in W_2^{2m,m}(\mathbb{R}^{n+1})$ , we have  $g \in \dot{H}^s(\mathbb{R}^{n+1})$  and*

$$\|g\|_{\dot{H}^s} \leq \|f\|_{W_2^{2m,m}}. \tag{5.5.1}$$

The norm in the homogeneous Sobolev space  $\dot{H}^s$  is given by  $\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^{n+1}} \|\xi\|^{2s} |\hat{f}(\xi)|^2 d\xi$  where  $\|\cdot\|$  is the usual Euclidean distance.

**Proof.** Follows directly since  $1 \leq s \leq m$ , using the definition of the norm in  $\dot{H}^s$ . □

**Lemma 5.2** *Under the same hypothesis of Theorem 1.5, we have:*

$$\|g\|_{L^\infty} \leq C \left[ 1 + \|f\|_{W_2^{2m,m}} \left( \log \left( e + \frac{\|f\|_{W_2^{2m,m}} + \|g\|_{L^\infty}}{\|f\|_{W_2^{2m,m}}} \right) \right)^{1/2} \right]. \tag{5.5.2}$$

**Proof.** We consider the isotropic ( $a = (1, \dots, 1)$ ) homogeneous dyadic partition of unity  $(\psi_j)_{j \in \mathbb{Z}}$  with  $\sum_{j \in \mathbb{Z}} \psi_j = 1$  and  $\hat{\varphi}_j = \psi_j$ . Fix some  $0 < \gamma < 1$ , and take an arbitrary  $N \in \mathbb{N}^*$ . We write:

$$\|g\|_{L^\infty} \leq \sum_{j \leq 0} \|\varphi_j * g\|_{L^\infty} + \sum_{j=1}^N \|\varphi_j * g\|_{L^\infty} + \sum_{j > N} \|\varphi_j * g\|_{L^\infty}. \tag{5.5.3}$$

We estimate the right-hand side of (5.5.3). Benstein’s inequality gives:

$$\|\varphi_j * g\|_{L^\infty} \leq C 2^{\left(\frac{n+1}{2}\right)j} \|\varphi_j * g\|_{L^2}. \tag{5.5.4}$$

We let  $s = \frac{n+1}{2}$ . Using (5.5.4), we compute:

$$\sum_{j \leq 0} \|\varphi_j * g\|_{L^\infty} \leq C \sum_{j \leq 0} 2^{sj} \|\varphi_j * g\|_{L^2} \leq \frac{C}{1 - 2^{-s}} \|g\|_{L^2}. \tag{5.5.5}$$

Again, using (5.5.4), we obtain:

$$\sum_{j=1}^N \|\varphi_j * g\|_{L^\infty} \leq C \sum_{j=1}^N 2^{sj} \|\varphi_j * g\|_{L^2} \leq CN^{1/2} \|g\|_{\dot{B}_{2,2}^s},$$

which, together with the fact that  $\dot{B}_{2,2}^s \simeq \dot{H}^s$ , and estimate (5.5.1) of Lemma 5.1, yield:

$$\sum_{j=1}^N \|\varphi_j * g\|_{L^\infty} \leq CN^{1/2} \|f\|_{W_2^{2m,m}}. \tag{5.5.6}$$

The last term of the right-hand side of (5.5.3) can be estimated as follows:

$$\sum_{j>N} \|\varphi_j * g\|_{L^\infty} = \sum_{j>N} 2^{-j\gamma} (2^{j\gamma} \|\varphi_j * g\|_{L^\infty}) \leq 2^{-\gamma N} \left( \frac{2^{-\gamma}}{1 - 2^{-\gamma}} \right) \|g\|_{\dot{B}_{\infty,\infty}^\gamma}. \tag{5.5.7}$$

We know that  $\dot{B}_{\infty,\infty}^\gamma \simeq \dot{C}^\gamma$ ; the homogeneous Hölder space whose semi-norm can be estimated as follows:

$$\|g\|_{\dot{C}^\gamma} = \sup_{z_1 \neq z_2} \frac{|g(z_1) - g(z_2)|}{\|z_1 - z_2\|^\gamma} \leq \|f\|_{W_2^{2m,m}} + \|g\|_{L^\infty}.$$

This, together with (5.5.7) yield:

$$\sum_{j>N} \|\varphi_j * g\|_{L^\infty} \leq C 2^{-\gamma N} (\|f\|_{W_2^{2m,m}} + \|g\|_{L^\infty}). \tag{5.5.8}$$

Combining (5.5.3), (5.5.5), (5.5.6) and (5.5.8), we finally get:

$$\|g\|_{L^\infty} \leq C(1 + N^{1/2} \|f\|_{W_2^{2m,m}} + 2^{-\gamma N} (\|f\|_{W_2^{2m,m}} + \|g\|_{L^\infty})).$$

By optimizing (as in Step 2 of Lemma 3.2) in  $N$  the above inequality, the proof easily follows. □

We are now ready to give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** As it was already mentioned in the beginning of this section, the proof relies on considering two cases.

**Case 1** ( $\|f\|_{W_2^{2m,m}} \leq 1$ ). Let  $A := \|g\|_{L^\infty}$ . Using inequalities (3.3.5) and (5.5.2), we obtain:

$$A \leq C[1 + (\log(e + 1 + A))^{1/2}],$$

which directly implies that  $A \leq C$ , and hence (1.1.11) is obtained.

**Case 2** ( $\|f\|_{W_2^{2m,m}} \geq 1$ ). Dividing inequality (5.5.2) by  $\|f\|_{W_2^{2m,m}}$ , we obtain:

$$\frac{\|g\|_{L^\infty}}{\|f\|_{W_2^{2m,m}}} \leq C \left[ 1 + \left( \log \left( e + 1 + \frac{\|g\|_{L^\infty}}{\|f\|_{W_2^{2m,m}}} \right) \right)^{1/2} \right].$$

Letting  $B := \|g\|_{L^\infty} / \|f\|_{W_2^{2m,m}}$ , we can easily see that  $B$  satisfies (as the term  $A$  in Case 1):

$$B \leq C[1 + (\log(e + 1 + B))^{1/2}],$$

which shows that  $B \leq C$ , and the proof is done. □

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