

Two Dimensional Unsteady State Non-homogeneous Thermoelastic Problem of Circular plate

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Abstract

This work concern with determination of two dimensional nonhomogeneous temperature distribution, displacement distribution and quasi-static thermal stresses due to internal heat generation at the rate $g(r, z, t)$. The circular plate is considered with arbitrary initial temperature and subjected to time dependent heat flux at the outer circular boundary $r = a$. The upper surface ($z = h$) of the circular plate is insulated and the lower surface ($z = 0$) of the plate is at zero temperature for any time $t > 0$, heat is generated within this circular plate at the rate $g(r, z, t)$. The governing heat conduction equation has been solved by integral transform technique. The results are obtained in series form in terms of Bessel's functions. The mathematical model has been constructed for copper material and the results for temperature, displacement and stresses have been computed numerically by using computational mathematical software MathCad 2007 and are depicted graphically by using Microsoft Office Excel 2007.

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1 Introduction

Plates are widely used in engineering applications such as aeronautical, naval and structural fields. They are made of homogeneous, non homogeneous, laminated or functionally graded materials in various shapes, sizes and thicknesses depending on their applications, depending on the plate thickness. In the classical plate theory, the first order and the higher order shear deformation plate theories, those are usually use to formulates the differential

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equation of plates. Nowaki[1] has determined the temperature distribution on upper face, with zero temperature on the lower surface and circular edge thermally insulated. Roy Choudhary[2] has studied the quasi-static thermal stresses due to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and the fix circular edge of the plate thermally insulated. Phythian[3] discuss the cylindrical heat flow with arbitrary heating rates at the outer surface and zero heat flux at the internal boundary. In the recent work, some problem have been solved by Noda et al.[4]. Gaikwad and Ghadle[5] recently obtain the thermal stresses of three dimensional non homogeneous thermoelastic rectangular plate. Recently Deshmukh et.al [6] and [7] determine the temperature distribution in hollow cylinder and the thermal stresses induced by a point heat source in a circular plate by quasi-static approach respectively.

In this paper the work of Deshmukh et. al [7] has been extended for two dimensional non-homogenous thermoelastic problem. We consider two dimensional non-homogeneous circular plate, this problem consist of determination of the temperature, displacement and quasi-static thermal stresses due to internal heat generation within it. Consider a thin circular plate of thickness h occupying the space D defined by $0 \leq r \leq a$, $0 \leq z \leq h$. Initially the plate is kept at arbitrary temperature $f(r, z)$. An arbitrary time dependent heat flux $Q(z, t)$ is applied on the outer surface ($r = a$). Also the upper surface ($z = h$) of the circular plate is insulated and the lower surface ($z = 0$) of the plate is at zero temperature for any time $t > 0$, heat is generated within this circular plate at the rate $g(r, z, t)$. The governing heat conduction equation has been solved by integral transform technique as in Ozisik[8]. The results are obtained in series form in terms of Bessel's functions. The results for temperature, displacement and stresses have been computed numerically and are illustrated graphically.

To the author's knowledge, no literature on two dimensional non-homogeneous thermoelastic problem in a circular plate due to internal heat generation has been published. The results presented here will be more useful in engineering problem particularly in aerospace engineering for stations of missiles body not influenced by hose tapering. The missile skill material assume to have physical properties independent of temperature, so that the temperature $T(r, z, t)$ is a function of radius, thickness and time only. Under these conditions, the displacement and thermal stresses in a thin circular plate due to heat generation is required to determined.

2 Mathematical Formulation

consider a thin circular plate occupying the space D defined by $0 \leq r \leq a, 0 \leq z \leq h$. The displacement equations of thermoelasticity have the form as;

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu} \right) e_{,i} = 2 \left(\frac{1+\nu}{1-\nu} \right) a_t T \quad (1)$$

$$e = U_{k,k}; \quad k, i = 1, 2. \quad (2)$$

where U_i -Displacement component, e -Dilation, T -Temperature, ν -Poisson ratio and a_t -Linear coefficient of thermal expansion of the material of the circular plate.

Introducing $\psi_i = U_i, \quad i = 1, 2.$

$$\nabla_1^2 \psi = (1+\nu)a_t T, \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (3)$$

$$\sigma_{ij} = 2\mu(\psi_{ij} - \delta_{ij}\psi_{,kk}), \quad i, j, k = 1, 2. \quad (4)$$

where μ -lame constant and δ_{ij} -is the Kroneker function, In axially symmetric case $\psi = \psi(r, z, t), T = T(r, z, t)$ and the differential equation governing the displacement potential function $\psi(r, z, t)$ is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = (1+\nu)a_t T \quad (5)$$

$$\text{with } \frac{\partial \psi}{\partial r} = 0 \text{ at } r = a, \text{ for all timet} \quad (6)$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r} \frac{\partial \psi}{\partial r} \quad (7)$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \psi}{\partial r^2} \quad (8)$$

The surface of thin circular plate at $r = a$ is assumed to be traction free. The boundary condition can be taken as

$$\sigma_{rr} = 0 \text{ at } r = a \quad (9)$$

Also in the plane state of stress within the plate

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \quad (10)$$

Initially

$$T = \psi = \sigma_{rr} = \sigma_{\theta\theta} = f(r, z) \quad (11)$$

The temperature of the thin circular plate $T(r, z, t)$ at a time t satisfies the differential equation given by ozisik(1968)[8] as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r, z, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (12)$$

with boundary conditions

$$\frac{\partial T}{\partial r} = Q(z, t) \text{ at } r = a, t > 0 \quad (13)$$

$$T = 0 \text{ at } z = 0, t > 0 \quad (14)$$

$$\frac{\partial T}{\partial z} = 0 \text{ at } z = h, t > 0 \quad (15)$$

and initial condition

$$T(r, z, t) = F(r, z) \text{ in } 0 \leq r \leq a, 0 \leq z \leq h, \text{ for } t = 0 \quad (16)$$

where k and α are the thermal conductivity and thermal diffusivity of the material of the circular respectively. Equations (1) to (16) constitute the mathematical formulation of the thermoelastic problem in a circular plate.

3 The Solution

3.1 Determination of $T(r, z, t)$

To obtain the expression for temperature $T(r, z, t)$ we develop the finite Fourier transform, the finite Hankel transform and their inversion and operating on the heat conduction equation (12) and (16), one obtains the expression for temperature as

$$\begin{aligned} T(r, z, t) = & \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} K(\eta_p, z) K_0(\beta_m, r) e^{-\alpha(\beta_m^2 + \eta_p^2)t} \\ & \times \left\{ \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') f(r', z') dr' dz' + \int_{t'=0}^t e^{\alpha(\beta_m^2 + \eta_p^2)t'} \right. \\ & \times \left[\frac{\alpha}{k} \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' \right. \\ & \left. \left. + \alpha a K_0(\beta_m, a) \int_{z'=0}^h K(\eta_p, z') Q(z', t') dz' \right] dt' \right\} \end{aligned} \quad (17)$$

where $K(\eta_p, z) = \sqrt{\frac{2}{h}} \sin(\eta_p, z)$, and η_1, η_2, \dots are the positive roots of the transcendental equation

$$\cos(\eta_p, z) = 0, \quad p = 1, 2, \dots \tag{18}$$

$$K_0(\beta_m, r) = \frac{\sqrt{2} J_0(\beta_m r)}{a J_0(\beta_m a)} \tag{19}$$

where β_1, β_2, \dots are the positive roots of the transcendental equation

$$J_1(\beta_m a) = 0 \tag{20}$$

3.2 Determination of Displacement function ψ

Using the equation (17) in equation (5) and making use of well known relation $\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] J_0(\beta_n r) = -\beta_n^2 J_0(\beta_n r)$, one obtain the displacement function $\psi(r, z, t)$ as

$$\begin{aligned} \psi(r, z, t) = & -(1 + \nu) a_t \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} K(\eta_p, z) \frac{K_0(\beta_m, r)}{\beta_m^2} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \\ & \times \left\{ \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') f(r', z') dr' dz' + \int_{t'=0}^t e^{\alpha(\beta_m^2 + \eta_p^2)t'} \right. \\ & \times \left[\frac{\alpha}{k} \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' \right. \\ & \left. \left. + \alpha a K_0(\beta_m, a) \int_{z'=0}^h K(\eta_p, z') Q(z', t') dz' \right] dt' \right\} \end{aligned} \tag{21}$$

3.3 Determination of Thermal Stresses σ_{rr} and $\sigma_{\theta\theta}$

Substituting the equation (21) in to equations (7) and (8), one obtain the thermal stresses σ_{rr} and $\sigma_{\theta\theta}$ as

$$\begin{aligned} \sigma_{rr} = & 2\mu(1 + \nu) a_t \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} K(\eta_p, z) \frac{K_1(\beta_m, r)}{r \beta_m^2} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \\ & \times \left\{ \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') f(r', z') dr' dz' + \int_{t'=0}^t e^{\alpha(\beta_m^2 + \eta_p^2)t'} \right. \\ & \times \left[\frac{\alpha}{k} \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' \right. \\ & \left. \left. + \alpha a K_0(\beta_m, a) \int_{z'=0}^h K(\eta_p, z') Q(z', t') dz' \right] dt' \right\} \end{aligned} \tag{22}$$

and

$$\begin{aligned}
 \sigma_{\theta\theta} = & 2\mu(1 + \nu)a_t \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} K(\eta_p, z) \frac{K_2(\beta_m, r)}{\beta_m^2} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \\
 & \times \left\{ \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') f(r', z') dr' dz' + \int_{t'=0}^t e^{\alpha(\beta_m^2 + \eta_p^2)t'} \right. \\
 & \times \left[\frac{\alpha}{k} \int_{r'=0}^a \int_{z'=0}^h r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t') dr' dz' \right. \\
 & \left. \left. + \alpha a K_0(\beta_m, a) \int_{z'=0}^h K(\eta_p, z') Q(z', t') dz' \right] dt' \right\} \quad (23)
 \end{aligned}$$

where $K_1(\beta_m, r) = \frac{\sqrt{2} \beta_m J_1(\beta_m r)}{a J_0(\beta_m a)}$ and

$$K_2(\beta_m, r) = \frac{\sqrt{2} \beta_m}{a J_0(\beta_m a)} \left[J_0(\beta_m r) - \frac{J_1(\beta_m r)}{\beta_m r} \right]$$

4 Special Case and Numerical Results

Setting $F(r, z) = r^2 \times (z^2 - h^2)^2 \times z^2$

$$g(r, z, t) = g_{pi} \delta(r - r_1) \delta(z - z_1) \delta(t - \tau)$$

$$Q(z, t) = z^2 \times (z^2 - h^2)^2 \times e^{-\omega t}, \alpha > 0$$

where r is the radius measured in meter, δ is the Dirac-delta function, $\omega = 10$, $g_{pi} = 50J/m$, $t \rightarrow \tau = 5$.

- Radius of circular plate $a = 1$ m
- Central path of circular plate in radial direction $r_1 = 0.5$ m
- Central path of circular plate in axial direction $z_1 = 0.1$ m

The numerical calculation has been carried out for a Copper(pure) circular plate with the material properties defined as

- Thermal diffusivity $\alpha = 112.34 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$.
- Thermal conductivity $k = 386 \text{ w/mk}$.
- Density $\rho = 8954 \text{ Kg/m}^3$.
- Specific heat $c_p = 383 \text{ J/kgK}$.
- Poisson ratio $\nu = 0.35$.
- Coefficient of linear thermal expansion $a_t = 16.5 \times 10^{-6} 1/K$.

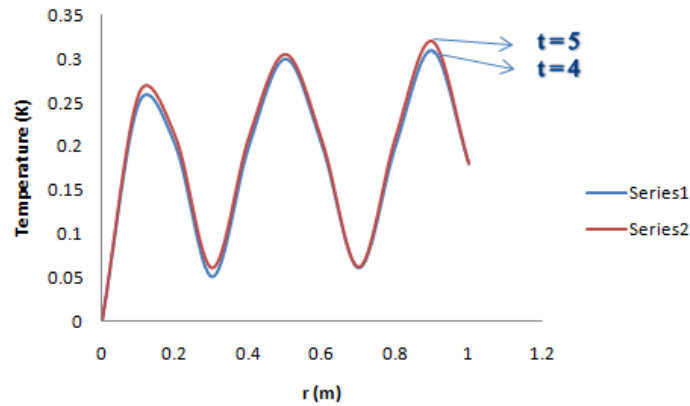


Figure 1: Temperature distribution T/A

- Lamé constant $\mu = 26.67$.
- $\beta_1 = 3.8317, \beta_2 = 7.0156, \beta_3 = 10.1735, \beta_4 = 13.3237$, and $\beta_5 = 16.470$ are the positive roots of transcendental equation (20)

We set for convenience

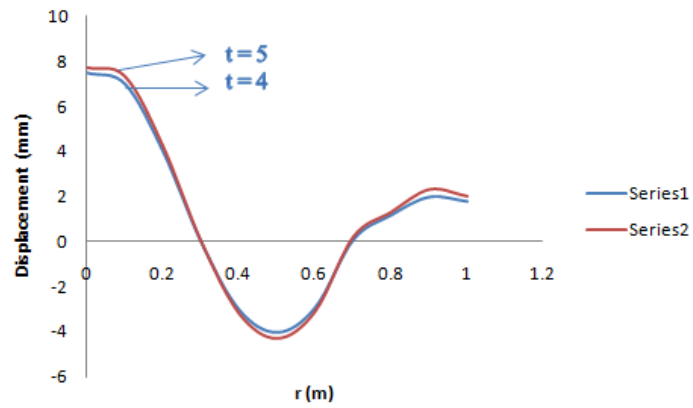
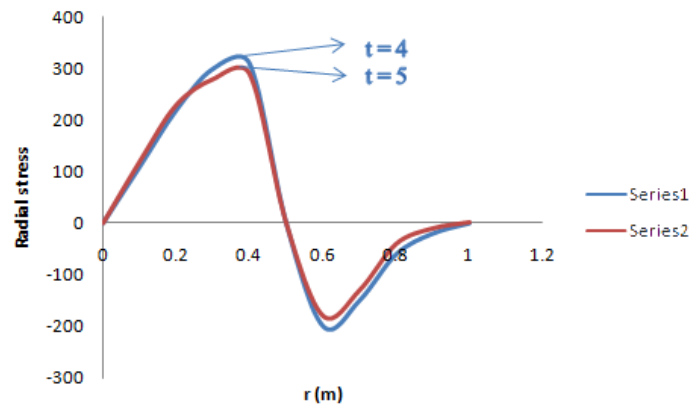
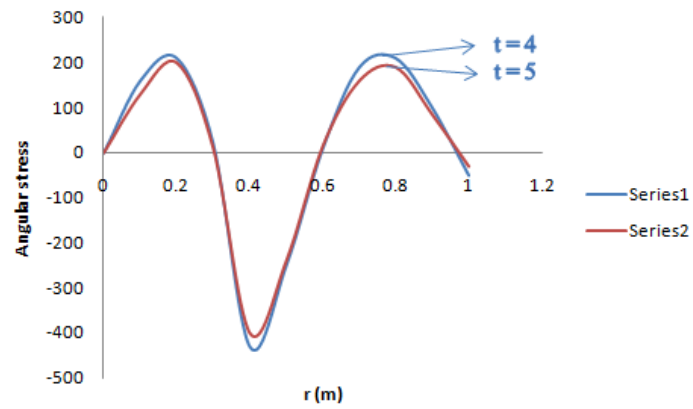
$$A = \frac{\sqrt{2}}{100h}, \quad B = \frac{\sqrt{2}(1+\nu)a_t}{100h}, \quad C = \frac{2\sqrt{2}(1+\nu)\mu a_t}{100h}$$

here A, B and C are constants. The numerical calculation has been carried out with the help of computational mathematical software MathCad 2007 and the graphs are plotted with the help of Microsoft Excel2007.

5 Conclusion

- From Fig.1, it is observed that, the temperature distribution the analytical behavior in the the form of wave due to internal point heat source.
- From Fig.2, it is observed that the displacement varies non-uniformly in the radial direction due to point heat source.
- From Fig.3, it is observed that the radial stresses develop due to an instantaneous point heat source in the radial direction in circular plate.
- In Fig.4, it is seen that the angular stresses develop the compressive stresses in the axial direction

In this paper, we extend the work of [7] to two dimensional non-homogeneous boundary value problem of heat conduction in a circular plate and determined the expressions of temperature, displacement and stresses due to internal heat generation. As a special case, a mathematical model is constructed for copper

Figure 2: Displacement distribution ψ/B Figure 3: Radial stress distribution σ_{rr}/C Figure 4: Angular stress distribution $\sigma_{\theta\theta}/C$

(pure) circular plate with specified material properties. Due to internal heat generation within the circular plate the radial stress develops as compressive stresses, whereas the angular stress develops with compressive stresses around the center and tensile stresses around the outer circular edge. Also it can be observed from the figures of temperature reaches maximum at central part of a circular plate .

The results obtained here are useful for engineering problems, particularly in the determination of the state of stress in thin circular plates. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in equations (17),(21)-(23).

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