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Convergence Theorems for the Class of Zamfirescu Operators

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Abstract

In this paper, we establish two strong convergence theorems to approximate fixed points of Zamfirescu operators in normed linear spaces. In the first part of the paper the generalised Mann iteration scheme is used to establish a strong convergence theorem. Our result generalizes and improves upon, among others, the corresponding result of Vasile Berinde [2]. A new two step iteration scheme is introduced in the second part of the paper and a strong convergence theorem is proved for the class of Zamfirescu operators. We observe that our result extends the corresponding result obtained by Isa Yildirim in [10].

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1 Introduction and preliminary definitions

The study of fixed points of various classes of mappings using different iterative schemes have been the focus of vigorous research for many mathematcians. In the last five decades many papers have been published on the iterative approximation of fixed points for various classes of operators using several iteration processes such as Krasnoselskij, Mann and Ishikawa iteration method, Mann and Ishikawa iteration process with errors and so on. In 2003, O. O. Owojori [5] proved a convergence theorem to approximate fixed points of Pseudo contrative mappings by the generalised Mann iteration in arbitrary real Banach spaces. In this paper we use the generalised Mann iteration scheme to establish a convergence theorem to approximate fixed points of Zamfirescu operators. Recently Thianwan [9] introduced a new two step iteration scheme to approximate common fixed points for two asymptotically nonexpansive nonself mappings in Banach spaces. Inspired by this fact we introduced a new two step iteration scheme to approximate fixed points of Zamfirescu operators.

The generalised Mann iteration scheme is defined as follows.

Let X be a normed linear space, K be nonempty, closed, convex subset of X. Let $T : K \to K$ and $S : K \to K$ are two mappings. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined for arbitrary $x_0 \in K$ as

$$x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n, \quad n = 0, 1, 2......... \tag{1}$$

where $\{a_n\}, \{b_n\}\, and \{c_n\}$ are real sequences in $[0, 1]$ with
$a_n + b_n + c_n = 1, b_n + c_n = \alpha_n$

We note that when $T = S$ the iteration (1) reduces to Mann iteration.

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \qquad n = 0, 1, 2... \tag{2}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.
For $\alpha_n = \lambda$(constant) the iteration (2) reduces to the so called Krasnoselskij iteration, while for $\alpha_n = 1$ we obtain the Picard iteration or method of successive approximation as it is commonly known, see [1 p.16].

The new two step iteration scheme is defined as follows.

Let K be a nonempty closed convex subset of a normed linear space X.
Let $T : K \to K$ and $S : K \to K$ be two nonlinear operators. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined for arbitrary $x_o \in K$ as

$$x_{n+1} = a_n y_n + b_n T y_n + c_n S y_n \tag{3}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n$$

where $\{a_n\}$,$\{b_n\}$,$\{c_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$
with $a_n + b_n + c_n = 1, b_n + c_n = \alpha_n$

We note that when $T = S$ the iteration(3) reduces to the iteration given by Thianwan [9],

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n \tag{4}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n = 0, 1, 2, ...$$

where $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$.
When $\beta_n = 0$ the iteration (4) reduces to the Mann iteration (2).

Consequently for $\alpha_n = \lambda$(constant) and $\alpha_n = 1$ we get the Krasnoselskij iteration and the Picard iteration respectively.

We use the following definitions in a metric space (X, d):

A mapping $T : X \to X$ is called an a-contraction if
(z_1) $\quad d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, where $a \in [0, 1)$.

The map T is called a Kannan mapping[4] if there exists $b \in [0, \frac{1}{2})$ such that
(z_2) $\quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$.

A similar definition is due to Chatterjea[3] : there exists $c \in [0, \frac{1}{2})$ such that
(z_3) $\quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

It is known [6] that $(z_1), (z_2), and (z_3)$ are independent contractive conditions. An operator T which satisfies at least one of the contractive conditions $(z_1), (z_2)$ and (z_3) is called a Zamfirescu operator or a Z-operator[11].
The class of Zamfirescu operators is one of the most studied class of quasi-contractive type operators. Zamfirescu showed in [11] that an operator satisfying condition Z has a unique fixed point that can be approximated using Picard iteration. Later Rhoads [7], [8] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operators. The following result was obtained by Rhoades [7, Theorem 4].

Theorem 1.1. *Let E be a uniformly convex Banach space, K be a closed, convex subset of E. Let $T : K \to K$ be a Zamfirescu operator.Let $\{x_n\}_{n=o}^{\infty}$ be defined by (2) for arbitary $x_0 \in K$ with $\{\alpha_n\}$ satisfying (1) $\alpha_0 = 1$; (2) $0 < \alpha_n < 1$ for $n \geq 1$; $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ .Then $\{x_n\}_{n=o}^{\infty}$ converges strongly to a fixed point of T.*

Berinde[2] extended the above theorem to arbitrary Banach spaces and also to Mann iteration by using weaker assumptions on the sequence α_n as follows.

Theorem 1.2. *Let E be an arbitrary Banach space, K be a closed, convex subset of E. Let $T : K \to K$ be a Zamfirescu operator. Let $\{x_n\}_{n=o}^{\infty}$ be defined by (2) for arbitary $x_0 \in K$ with $\{\alpha_n\} \in [0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=o}^{\infty}$ converges strongly to a fixed point of T.*

Our result generalises the result in Berinde [2]

We need the following lemma:

Lemma 1.3. [1, p.13]. *Let $\{a_n\}, \{b_n\}$ and $\{t_n\}$ be the sequences of non-negative numbers satisfying*

$a_{n+1} \leq (1-\omega_n)a_n + b_n + t_n \qquad \forall n \geq 0$ *where* $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$.
If $\sum_{n=0}^{\infty} \omega_n = \infty, b_n = O(\omega_n)$ *and* $\sum_{n=0}^{\infty} t_n < \infty$ *then* $\lim_{n\to\infty} a_n = 0$

2 Main results

Theorem 2.1. *Let K be a nonempty, closed, convex subset of a normed space X. Let $T : K \to K$ be a Zamfirescu operator with $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T ,$S : K \to K$ be continuous and $(S-T)$is bounded. Let $\{x_n\}_{n=0}^{\infty}$ be the iteration for arbitrary $x_0 \in K$ defined as in (1) where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in $[0,1]$ with $a_n + b_n + c_n = 1, b_n + c_n = \alpha_n$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of T.*

Proof. From $F(T) \neq \phi$,we get that T has atleast one fixed point in K say, x^*
Since T is a Zamfiresu operator, at least one of conditions from $(z_1), (z_2), (z_3)$ is satisfied.
If z_2 holds, then

$$\begin{aligned} \|Tx - Ty\| &\leq b\left[\|x - Tx\| + \|y - Ty\|\right] \\ &\leq b\left[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right] \end{aligned}$$

which implies that

$$(1-b)\|Tx - Ty\| \leq b\|x - y\| + 2b\|x - Tx\|.$$

Since $0 \leq b < 1/2$ we get

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|. \tag{5}$$

Similarly if (z_3) holds we obtain

$$\|Tx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\|. \tag{6}$$

$$Let \qquad \delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{7}$$

Then $0 \leq \delta < 1$ and from (5), (6), (7) we get the inequality,

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \qquad \forall x, y \in K. \tag{8}$$

Let $\{x_n\}_{n=0}^{\infty}$ be the iteration defined by (1) and $x_0 \in K$ be arbitrary. Then

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n x_n + b_n T x_n + c_n S x_n - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + (\alpha_n - c_n) T x_n + c_n S x_n - \alpha_n x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T x_n - x^*) + c_n (S - T) x_n\| . \end{aligned} \tag{9}$$

Taking $x = x^*, y = x_n$ in (8) we get

$$\|Tx^* - Tx_n\| \leq \delta \|x^* - x_n\| + 2\delta \|x^* - Tx^*\|$$

which implies

$$\|Tx_n - x^*\| \leq \delta \|x_n - x^*\| . \tag{10}$$

Now from (9) and (10) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n) \|x_n - x^*\| + \alpha_n \delta \|x_n - x^*\| + c_n \|(S-T)x_n\| \\ &= [1 - \alpha_n(1-\delta)] \|x_n - x^*\| + c_n \|(S-T)x_n\|. \end{aligned}$$

Since $0 \leq \delta < 1, \alpha_n \in [0,1], \sum_{n=0}^{\infty} \alpha_n = \infty$ and setting $a_n = \|x_n - x^*\|, \omega_n = \alpha_n(1-\delta)$ and by applying lemma 1.1, we get

$$\lim_{n \to \infty} \|x_{n+1} - x^*\| = 0$$

and hence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point $x^*$of T.

When $T = S$ Berinde's result [2, Theorem 2.1] is a corollary to our result.

Corollary 2.2. *Let K be a nonempty, closed, convex subset of a normed space X. Let $T : K \to K$ be a Zamfirescu operator with $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T. Let $\{x_n\}_{n=o}^{\infty}$ be defined by (2) for arbitary $x_0 \in K$ with $\{\alpha_n\} \in [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=o}^{\infty}$ converges strongly to a fixed point of T.*

Theorem 2.3. *Let K be a nonempty closed convex subset of a normed space X. Let $T : K \to K$ and $S : K \to K$ be two Zamfiresu operators with a common fixed point in K. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence for arbitrary $x_o \in K$ defined as in (3) where $\{a_n\}$,$\{b_n\}$,$\{c_n\}$ and $\{\beta_n\}$are real sequences in $[0,1]$ with*

$a_n + b_n + c_n = 1, b_n + c_n = \alpha_n$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the common fixed point of T and S.

Proof. Let the common fixed point of T and S be x^*. Since T is a Zamfiresu operator, proceeding with the arguments similar to those in the proof of theorem 2.1, we get the following inequality,

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \qquad \forall x, y \in K. \tag{11}$$

Similarly since S is a Zamfirescu operator, we get

$$\|Sx - Sy\| \leq \delta \|x - y\| + 2\delta \|x - Sx\| \qquad \forall x, y \in K. \tag{12}$$

Let $\{x_n\}_{n=0}^{\infty}$ be the sequences defined by (3). Then

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n y_n + b_n T y_n + c_n S y_n - x^*\| \\ &= \|(1-\alpha_n) y_n + b_n T y_n + c_n S y_n - (a_n + b_n + c_n) x^*\| \\ &= \|(1-\alpha_n) y_n - (1-\alpha_n) x^* + b_n (T y_n - x^*) + c_n (S y_n - x^*)\| \\ &= \|(1-\alpha_n)(y_n - x^*) + b_n (T y_n - x^*) + c_n (S y_n - x^*)\| . \end{aligned} \tag{13}$$

Put $x = x^*$ and $y = y_n$ in (11) we get

$$\|Ty_n - x^*\| \leq \delta \|y_n - x^*\| . \tag{14}$$

Put $x = x^*$ and $y = y_n$ in (12) we get

$$\|Sy_n - x^*\| \leq \delta \|y_n - x^*\| . \tag{15}$$

Put (14) and (15) in (13) we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n) \|y_n - x^*\| + b_n \delta \|y_n - x^*\| + c_n \delta \|y_n - x^*\| \\ &\leq [1 - \alpha_n + b_n \delta + c_n \delta] \|y_n - x^*\| \\ &\leq [1 - \alpha_n (1-\delta)] \|y_n - x^*\| . \end{aligned} \tag{16}$$

Now,

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\beta_n) x_n + \beta_n T x_n - x^*\| \\ &\leq (1-\beta_n) \|x_n - x^*\| + \beta_n \|T x_n - x^*\| \\ &\leq (1-\beta_n) \|x_n - x^*\| + \beta_n \delta \|x_n - x^*\| \\ &\leq (1 - \beta_n + \beta_n \delta) \|x_n - x^*\| \\ &\leq [1 - \beta_n (1-\delta)] \|x_n - x^*\| . \end{aligned} \tag{17}$$

Put (17) in (16) we get,

$$\|x_{n+1} - x^*\| \leq [1 - \alpha_n(1-\delta)]\,[1 - \beta_n(1-\delta)]\,\|x_n - x^*\|\,. \qquad (18)$$

ie,

$$\|x_{n+1} - x^*\| \leq [1 - \alpha_n(1-\delta)]\,\|x_n - x^*\|\,.$$

Since $0 \leq \delta < 1, \alpha_n \in [0,1], \sum_{n=0}^{\infty} \alpha_n = \infty$ and setting $a_n = \|x_n - x^*\|$, $\omega_n = \alpha_n(1-\delta)$ by lemma 1.2 we get

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

Hence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T and S. When $T = S$ the result in Isa Yildirim [10, Theorem 2.1] is a corollary to our result.

Corollary 2.4. *Let K be a nonempty,closed,convex subset of a normed space X. Let $T : K \to K$ be Zamfirescu operator with $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T. Let $\{x_n\}_{n=o}^{\infty}$ be the sequence defined by (4) where $\{\alpha_n\}$and $\beta_n \in [0,1]$ satisfying$\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=o}^{\infty}$ converges strongly to a fixed point of T.*

Remarks

1. Since the Kannan's and Chattejea's contractive conditions are both included in the class of Zamfirescu operators, by Theorem 2.1 and 2.3 we obtain corresponding convergence theorems in these classes of operators.

2. Our main results also include the convergence of both Picard and Krasnoselskij iterations as a particular case.

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