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Regular Semigroups with a S^0- Orthodox Transversal

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Abstract

In this paper, we consider another generalization for quasi-ideal orthodox transversal, the so-called S^0-orthodox transversals. We give a structure theorem for regular semigroups with S^0-orthodox transversals. If S^0 is a S^0-orthodox transversal of S then S can be described in terms of S^0.

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1 Introduction

Let S be a regular semigroup and S^0 be a regular subsemigroup of S. A natural question that has been considered by many authors is to what extent is S determined by S^0? The concept of an inverse transversal is one of the answer to this question. Recall that an inverse transversal of a regular semigroup S is an inverse subsemigroup S^0 that contains precisely one inverse for every $x \in S$. In 1982, Blyth and McFadden introduced the class of regular semigroups with an inverse transversal [1].

Recently, the concept of inverse transversal was generalized by many authors [2-10]. In particular, the concept of orthodox transversals was introduced by Chen Jianfei [2] as a generalization of inverse transversals. Chen Jianfei obtained an excellent structure theorem for regular semigroups with quasi-ideal orthodox transversals. In 2007, Xiangjun Kong [7] constructed regular semigroups with quasi-ideal orthodox transversals by a simpler format set. In 2009, Xiangjun Kong and Xianzhong Zhao [10] gave a structure theorem for regular semigroups with quasi-ideal orthodox transversals by two orthodox semigroups. Hence the general case of orthodox transversals is to be considered. The main results are the sets

$$I = \{aa^0 : a \in S,\ a^0 \in V(a) \cap S^0\}$$

and

$$\Lambda = \{a^0 a : a \in S,\ a^0 \in V(a) \cap S^0\}$$

are two components of regular semigroups with orthodox transversals. Chen-Jianfei [2] have shown that I and Λ are subbands if S^0 is a quasi-ideal orthodox transversal of S. Though each element of the sets I and Λ is an idempotent, they are necessarily subbands of S. In 2001, Chen Jianfei and Guo Yugi [3] shown that, if S^0 is an orthodox transversal of S, then the semi bands $\langle I \rangle$ and $\langle \Lambda \rangle$ generated by I and Λ respectively are bands. In this paper, we consider another generalization for quasi-ideal orthodox transversal, called S^0- orthodox transversals. We give a structure theorem for regular semigroups with S^0-orthodox transversals. This is also one of the answer to our question. That is, if S^0 is a S^0- orthodox transversal of S then S can be described in terms of S^0.

Section 2 presents some necessary notation and known results.In section 3, we introduce two new subclasses, $S-$ orthodox transversals and S^0- orthodox transversals of orthodox transversals and we obtain some basic properties of I and Λ when S is an S^0- orthodox transversal. In section 4, we give a structure theorem for regular semigroup with S^0- orthodox transversals. When S^0 is a quasi ideal of S, our theorem simplifies considerably.

2 Preliminaries

We adopt the terminology, notation and results of [2] and [3].

Definition 2.1 Let S be a semigroup and S^0 a subsemigroup of S. We call S^0 an *orthodox transversal* of S if the following conditions are satisfied.

(i) $V_{S^0}(x) \neq \phi$ for all $x \in S$.
(ii) if $x, y \in S$ and $\{x, y\} \cap S^0 \neq \phi$, then $V_{S^0}(x)V_{S^0}(y) \subseteq V_{S^0}(yx)$.

Note that if S^0 is an orthodox transversal of S, then S is regular by (i) and S^0 is an orthodox subsemigroup of S by (ii).

Theorem 2.2 *Let S be a regular semigroup and S^0 a quasi-ideal orthodox transversal of S. Then*

(i) $I \cap \Lambda = E(S^0)$
(ii) $I = \{e \in E(S) : (\exists e^* \in E(S^0)), e\mathrm{L}e^*\}$
$\Lambda = \{f \in E(S) : (\exists f^+ \in E(S^0)), f\mathrm{R}f^+\}$
(iii) $IE(S^0) \subseteq I, \ E(S^0)\Lambda \subseteq \Lambda$.
(iv) I *and* Λ *are subbands of* S.

Theorem 2.3 *Let S^0 be a quasi-ideal orthodox transversal of a regular semigroup S. Then*

(i) if $e \in I$ (or Λ) then $V_{S^0}(e) \subseteq E(S^0)$.
(ii) if $x \in S$ and $x^0 \in V_{S^0}(x)$, then $V_{S^0}(x) = V_{S^0}(x^0x)x^0V_{S^0}(xx^0)$.
(iii) if $V_{S^0}(x) \cap V_{S^0}(y) \neq \phi$ for any $x, y \in S$, then $V_{S^0}(x) = V_{S^0}(y)$.

Theorem 2.4 *Let S^0 be an orthodox transversal of S, then the Green relation H on S saturates S^0. (That is, S^0 is a union of some $\mathrm{H}-$classes on S.) In particular, the maximum idempotent-separating congruence on S saturates S^0.*

Theorem 2.5 *Let S^0 be an orthodox transversal of S. Then S is an orthodox semigroup if and only if for every $a, b \in S$, $V_{S^0}(a)V_{S^0}(b) \subseteq V_{S^0}(ba)$.*

Lemma.2.6 *If S^0 is an orthodox transversal of S then for any $a, b \in S^0$,*
$V(a) \cap V(b) \neq \phi \Rightarrow V_{S^0}(a) = V_{S^0}(b)$.

Lemma.2.7 *Let S^0 be an orthodox transversal of S. For $e \in S$, if $V_{S^0}(e) \cap E(S^0) \neq \phi$, then $V_{S^0}(e) \subseteq E(S^0)$.*

Theorem.2.8 *Let S^0 be an orthodox transversal of S. The semiband $\langle I \rangle$ (respectively $\langle \Lambda \rangle$) generated by I (respectively Λ) is a subband of S.*

Note that if S^0 is an orthodox transversal of S then I is a band if and only if $E(S^0)I \subseteq I$.

3 S^0- ORTHODOX TRANSVERSALS

Definition 3.1 Let S^0 be an orthodox transversal of S. S^0 is said to be an $S-$*orthodox transversal* of S if I and Λ are subbands of S.

Definition 3.2 Let S^0 be an $S-$orthodox transversal of S. Then S^0 is said to be an S^0-*orthodox transversal* of S if the regular semigroup S^0SS^0 is an orthodox transversal of S.

It is clear that a quasi-ideal orthodox transversal is an S^0-orthodox transversal. We denote S^0SS^0 by U.

Lemma 3.3 *Let S^0 be an S^0-orthodox transversal of S. If $i \in I$, then $i\mathrm{R}e$ for some $e \in E(U)$ implies $i \in E(U)$; if $\lambda \in \Lambda$, then $\lambda\mathrm{L}e$ for some $e \in E(U)$ implies $\lambda \in E(U)$.*

Proof. If $i\mathrm{R}e$ then $i = ei = eii^* \in S^0SS^0 = U$, and hence $i \in E(U)$. The second statement can be proved dually.

Lemma 3.4 *Let S^0 be an S^0-orthodox transversal of S. If $i \in I$ $(or \Lambda)$ then $V_{S^0}(i) \subseteq E(U)$.*

Proof. Let $i \in I$. Take $i^* \in E(S^0)$ such that $i\mathrm{L}i^*$, and suppose that $x \in V_{S^0}(i)$ and $x^0 \in V_{S^0}(x)$. Since $x^0x \in E(S^0)$, $x^0xi \in S^0SS^0 = U$ and hence $x^0xi \in E(U)$. On the other hand, $i^*xx^0 \in E(U)$ since U is orthodox. Therefore,

$$\begin{aligned} x^0 &= x^0xx^0 \\ &= x^0xixx^0 \text{ since } x \in V_{S^0}(i) \\ &= x^0xi.i^*xx^0 \text{ since } i\mathrm{L}i^* \\ &= E(U).E(U) \subseteq E(U). \end{aligned}$$

Therefore, $x \in E(U)$, since U is orthodox. Thus $V_{S^0}(i) \subseteq E(U)$.

Define

$$\begin{aligned} \bar{I} &= \{i \in E(S) : (\exists i^* \in E(U)) : \imath^*\mathrm{L}i\} \\ \bar{\Lambda} &= \{\lambda \in E(S) : (\exists \lambda' \in E(U)) : \lambda'\mathrm{R}\lambda\} \end{aligned}$$

Clearly $\bar{I}$ and $\bar{\Lambda}$ are subbands of S.

For each $e \in E(U)$, let

$$\begin{aligned} I_e &= \{i \in \bar{I} : (\exists i^* \in E(U))\, i^*\mathrm{R}e\} \\ \Lambda_e &= \{\lambda \in \bar{\Lambda} : (\exists \lambda' \in E(U))\, \lambda'\mathrm{L}e\}. \end{aligned}$$

Lemma 3.5 *Let S^0 be an S^0-orthodox transversal of S. Then I_e and Λ_e are rectangular bands.*

Proof. Let $i, i_1 \in I_e$. Then there exist $i^*, i_1^* \in E(U)$ such that $i^*\mathrm{R}e\mathrm{R}i_1^*$. Since S^0 is S^0-orthodox transversal, $ii_1 \in I$ and $i^*i_1^* \in E(U)$. Further, since $i^*\mathrm{R}e\mathrm{R}i_1^*$, $i^*i_1^* = i_1^*\mathrm{R}e$. Hence I_e is a band. Let $i, i_1 \in I_e$. Then by Lemma 2.6, $V_{S^0}(ii_1i) = V_{S^0}(i)$. Since $E(U)$ is a band, so it is a semilattice of rectangular bands. Therefore ii_1i and i are in the same rectangular band. Hence

$$ii'i = i.ii'i.i = i.$$

Therefore I_e is a rectangular band. Similarly, Λ_e is also a rectangular band.

Lemma 3.6 *Let S^0 be an S^0-orthodox transversal of S. For any $i_1, i_2, i_3 \in \bar{I}$ with $i_3 \mathrm{R} i_1$, we have $i_3i_2 = i_1i_2$. Dually, for any $\lambda_1, \lambda_2, \lambda_3 \in \bar{\Lambda}$, with $\lambda_3 \mathrm{L} \lambda_1$, we have $\lambda_2\lambda_3 = \lambda_2\lambda_1$.*

Proof. If $i_1, i_2, i_3 \in \bar{I}$ then for some $i_1^*, i_2^*, i_3^* \in E(U)$, we have $i_1^* \mathrm{L} i_1$, $i_2^* \mathrm{L} i_2$ and $i_3^* \mathrm{L} i_3$. If $i_3 \mathrm{R} i_1$, then by Green's lemma, $i_3 i_1^* = i_1$, and hence $i_3 i_2 = i_3(i_1^* i_2) = (i_3 i_1^*) i_2 = i_1 i_2$. The second statement can be proved dually.

Maintaining the notation followed in [5], the following theorem is similar to Theorem of 2.5 of [5].

Theorem 3.7 *The association $r(e) \mapsto A_{r(e)}$, $(r(e), r(f)) \longmapsto A(r(e), r(f))$ where*

$$A_{r(e)} = I_e/R = \{\overrightarrow{i} \in \bar{I}/\mathrm{R} : (\exists i^* \in E(U))\, i^* \mathrm{R} e\}$$

with $\overrightarrow{e} = r(e)$ as base point and where the map

$$A(r(e), r(f)) : A_{r(e)} \to A_{r(f)}$$

is given by $\vec{i} A(r(e), r(f)) = \vec{ie}$, defines a functor $A : E(U)/\mathrm{R} \to P$.

Dually, the association, $\ell(e) \mapsto B_{\ell(e)}, (\ell(e), \ell(f)) \mapsto B(\ell(e), \ell(f))$, where $B_{\ell(e)} = \Lambda_e/\mathrm{L} = \{\overleftarrow{\lambda} \in \bar{\Lambda}/\mathrm{R} : (\exists \lambda' \in E(U))\ \lambda' \mathrm{L} e\}$ with $\overleftarrow{e} = \ell(e)$ as base point, and where the map $B(\ell(e), \ell(f)) : B_{\ell(e)} \to B_{\ell(f)}$ is given by $\overleftarrow{\lambda} B(\ell(e), \ell(f)) = \overleftarrow{f\lambda}$ defines a functor $B : E(U)/\mathrm{L} \to P$.

4 Main Theorem

Let us define S^0-pair for an orthodox semigroup S^0.

Definition 4.1 Let S^0 be an orthodox semigroup. By an S^0-*pair* (A, B) we mean a pair of functors

$$A : E(S^0)/\mathrm{R} \to P,\ B : E(S^0)/\mathrm{L} \to P.$$

Given an S^0 − pair (A, B), *a $B \times$ A matrix over* S^0 is a function

$$* : (b,a) \longmapsto b * a : \bigcup_{\ell(e)\in E(S^0)/\mathrm{L}} B_{\ell(e)} \times \bigcup_{r(f)\in E(S^0)/\mathrm{R}} A_{r(f)} \to S^0$$

Definition 4.2 Let (A, B) be an S^0-pair with a $B \times A$ matrix $*$ over S. By an *enrichment* $\xi = \xi(A, B)$ of (A, B) relative to $*$ we mean a family of maps

$$A^{x,y}_{b,a} : A_{r(x)} \to A_{r(x.b*a.y)}, \qquad B^{x,y}_{b,a} : B_{\ell(y)} \to B_{\ell(x.b*a.y)}$$

where $x, y \in S^0$, $b \in B_{\ell(x)}$, $a \in A_{r(y)}$, such that

(M_1) $A^{x,y}_{\ell(x),r(y)} = A(r(x), r(xy))$ and $B^{x,y}_{\ell(x),r(y)} = B(\ell(y), \ell(xy))$,

(M_2) if $x\mathrm{R}x.b * a.y$ then $A^{x,y}_{b,a} = id$; and if $y\mathrm{L}x.b * a.y$ then $B^{x,y}_{b,a} = id$,

(M_3) $A^{x,y}_{b,a} A^{x.b*a.y,z}_{c\,B^{x,y}_{b,a},d} = A^{x,y.c*d.z}_{b,a\,A^{y,z}_{c,d}}$,

(M_4) $B^{y,z}_{c,d} B^{x,y.c*d.z}_{b,a\,A^{y,z}_{c,d}} = B^{x.b*a.y,z}_{c\,B^{x,y}_{b,a},d}$,

(M_5) $x.b * a.y.c\, B^{x,y}_{b,a} * d.z = x.b * a\, A^{y,z}_{c,d}.y.c * d.z$

for all $x, y, z \in S$, $b \in B_{\ell(x)}$, $a \in A_{r(y)}$, $c \in B_{\ell(y)}$, $d \in A_{r(z)}$.

Theorem 4.3 *Let S^0 be an orthodox semigroup and let (A, B) be an S^0-pair. Let $*$ be a $B \times A$ matrix over S^0 satisfying*

(N_1) if $b \in B_{\ell(e)}$ and $a \in A_{r(f)}$ then $b * a \in \ell(e).S\,r(f)$.
(N_2) for any $b \in B_{\ell(e)}$, $a \in A_{r(f)}$, $b * r(f)$, $\ell(e) * a \in \ell(e)r(f)$.

Let ξ be an enrichment of (A, B) relative to $*$. Then the set

$$W = W(S^0; A, B; *; \xi) = \{(a, x, b) : x \in S^0, a \in A_{r(x)}, b \in B_{\ell(x)}\}$$

is a regular semigroup under the multiplication

$$(a, x, b)(c, y, d) = \{aA^{x,y}_{b,c},\ x.b{*}c.y,\ dB^{x,y}_{b,c}\} \tag{4.1}$$

The map $\eta : S^0 \to W$, $x\eta = (r(x), x\,\ell(x))$ is an injective homomorphism of S^0 to W. If we identify S^0 with $S^0\eta$, via η, then S^0 is an S^0-orthodox transversal of S.

Conversely, every regular semigroup with an S^0-orthodox transversal can be constructed in this way.

Proof. The associativity of the multiplication follows from $(M_3)-(M_5)$. We first prove that η is an injective homomorphism. Clearly η is one-to-one. Since $A^{x,y}_{\ell(x),r(y)}$, $B^{x,y}_{\ell(x),r(y)}$ are base point preserving function by (M_1), we get

$$\begin{aligned} x\eta.y\eta &= (r(x),x,\ell(x))\,(r(y),y,\ell(y)) \\ &= (r(xy),xy,\ell(xy)) \\ &= (xy)\eta. \end{aligned}$$

Hence η is an injective homomorphism.

Let $(a,x,b)\in W$. Then $x\in S^0$, let $x^*\in V_{S^0(x)}$, by (N_2) and (M_2),

$$(a,x,b)\,(r(x^*),x^*,\ell(x^*))\,(a,x,b) = (a,xx^*,\ell(x^*))(a,x,b) = (a,x,b)$$

and

$$\begin{aligned} &(r(x^*),x^*,\ell(x^*))\,(a,x,b)\,(r(x^*),x^*,\ell(x^*)) \\ &\qquad = (r(x^*),x^*x,b)(r(x^*),x^*,\ell(x^*)) \\ &\qquad = (r(x^*),x^*,\ell(x^*)) \end{aligned}$$

so that $(r(x^*),x^*,\ell(x^*))\in V_{S^0}((a,x,b))$, since we can identify S^0 with $S^0\eta$, $V_{S^0}((a,x,b))\neq\ \phi$.

Moreover, for any $(a,x,b)\in W$,

$$V_{S^0}((a,x,b)) = \{(r(x^*),x^*,\ell(x^*)) : x^*\in V_{S^0}(x)\}.$$

Hence W is a regular semigroup. Now let $(a,x,b)\in W$ and $r(y),y,\ell(y)\in S^0\eta\cong S^0$. Let $(r(x^*),x^*,\ell(x^*))\in V_{S^0}((a,x,b))$ and $(r(y^*),y^*,\ell(y^*))$ $\in V_{S^0}((r(y),y,\ell(y)))$. Then

$$(r(x^*),x^*,\ell(x^*))(r(y^*),y^*,\ell(y^*))\in V_{S^0}((a,x,b))V_{S^0}((r(y),y,\ell(y)))$$

$$\Rightarrow (r(x^*y^*),x^*y^*,\ell(x^*y^*))\in V_{S^0}((a,x,b))V_{S^0}((r(y),y,\ell(y))).$$

Consider $(r(y),y,\ell(y))\,(a,x,b) = (r(yx),yx,bB^{x,y}_{\ell(y),a})$ by (N_2) and $M_(2)$. Next we prove that $(r(x^*y^*),x^*y^*,\ell(x^*y^*))\in V_{S^0}((r(yx),yx,bB^{x,y}_{\ell(y),a}))$. But this is immediately follows, since S^0 is an orthodox semigroup and by (M_2) and (N_2).

Therefore, for any $(a,x,b)\in W$, $(r(y),y,\ell(y))\in S^0\eta\cong S^0$,

$$V_{S^0}((a,x,b))V_{S^0}((r(y),y,\ell(y)))\subseteq V_{S^0}((r(y),y,\ell(y))(a,x,b)).$$

Hence S^0 is an orthodox transversal.

Note that by (M_2),

$$E(W) = \{(a,x,b) \in W : x.(b*a).x = x\}.$$

Consider the sets

$$I = \{(a,x,b) \in E(W) : (\exists((r(x_1),x_1,\ell(x_1))) \in E(S^0)\,(a,x,b)\mathrm{L}(r(x_1),x_1,\ell(x_1))\}$$

$$\Lambda = \{(c,y,d) \in E(W) : (\exists((r(y_1),y_1,\ell(y_1))) \in E(S^0)\,(c,y,d)\mathrm{R}(r(y_1),y_1,\ell(y_1))\}.$$

Let $(a,x,b),(c,y,d) \in I$. Take $(r(x_1),x_1,\ell(x_1)) \in E(S^0)$ such that $(r(x_1),x_1,\ell(x_1))$ $\mathrm{L}(a,x,b)$, then

$$(a,x,b)(c,y,d) = (a,x,b)(r(x_1),x_1,\ell(x_1))(c,y,d)$$

But $(r(x_1),x_1,\ell(x_1)) \in E(S^0)$ by (N_2). Thus

$$(a,x,b)(c,y,d) = (a,x,b)(r(x_1),x_1,\ell(x_1)) \subseteq IE(S^0) \subseteq I,$$

by Theorem 2.2(iii). Hence I is a band. Similarly, we can prove Λ is a band. So S^0 is a $S-$orthodox transversal of S. Since S^0 is an orthodox transversal of S, S^0WS^0 is a regular subsemigroup of S. Let $(a,x,b) \in W$, $(r(x_1),x_1,\ell(x_1))$, $(r(x_2),x_2,\ell(x_2)) \in S^0\eta.W.S^0\eta \cong S^0WS^0$, then

$$(r(x_1),x_1,\ell(x_1))\,(a,x,b)(r(x_2),x_2,\ell(x_2)) = (r(m),m,\ell(m_1)) \text{ by } (N_2)$$

where $m = x_1.\ell(x_1)*a.x.b*r(x_2).x_2 \in S^0WS^0$.

Therefore,

$$\begin{aligned} U &= S^0WS^0 = S^0\eta.W.S^0\eta \\ &= \{(r(m),m,\ell(m)) : m \in S^0WS^0\}. \end{aligned}$$

By (N_2), U is an orthodox transversal of W. Let (A,B) be an $U-$pair with a $B \times A$ matrix $*$ over U and $\xi = \xi(A,B)$ be an enrichment of A,B relative to $*$.

Then the set

$$\overline{W} = \overline{W}(U;A,B,*,\xi) = \{((a,x,b) : x \in U,\, a \in A_{r(x)},\, b \in B_{\ell(x)}\}$$

is a regular semigroup under the multiplication given by (4.1) and U is an orthodox transversal of $\overline{W}$.

Conversely, suppose that S^0 is an S^0-orthodox transversal of S. Let (A, B) be the S^0SS^0- pair defined in Theorem 3.7, we define a $B \times A$ matrix * over $S^0SS^0 = U$ as follows. Fix an $R-$ invariant map $\alpha : \overline{I} \to \overline{I}$ so that α is constant on each R$-$ class of $\overline{I}$. Similarly fix an $L-$ invariant map $\beta : \overline{\Lambda} \to \overline{\Lambda}$ so that β is constant on each each L$-$ class of $\overline{\Lambda}$. For each $\overleftarrow{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)}$,

define
$$\overleftarrow{\lambda} * \overrightarrow{i} = (\lambda\beta)(i\alpha).$$

Clearly * is well defined. For, if $\overleftarrow{\lambda}_1 = \overleftarrow{\lambda}_2$, $\overrightarrow{i_1} = \overrightarrow{i_2}$ then $\lambda_1\beta = \lambda_2\beta$, $i_1\alpha = i_2\alpha$ and so $((\lambda\beta)(i\alpha)) = ((\lambda_2\beta)(i_2\alpha))$. We show that $*$ satisfies (N_1) and (N_2).

(N_1) If $\overleftarrow{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)}$, then
$$\begin{aligned}\overleftarrow{\lambda} * \overrightarrow{i} &= (\lambda\beta)(i\alpha)\\ &= e(\lambda\beta)(i\alpha)f \in \ell(e)Ur(f).\end{aligned}$$
(N_2) If $\overleftarrow{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)}$ then
$$\begin{aligned}\overleftarrow{\lambda} * r(f) &= (\lambda\beta)(f\alpha)\\ &= (\lambda\beta)^*(\lambda\beta)f(\alpha) \in \ell(e)r(f),\end{aligned}$$
since $(\lambda\beta) \in U = S^0SS^0$. Similarly, $\ell(e) * \overrightarrow{i} \in \ell(e)r(f)$.

For each quadruple $(x, y, \overrightarrow{i}, \overleftarrow{\lambda})$, where $x, y \in U$, $\overleftarrow{\lambda} \in B_{\ell(x)}$, $\overrightarrow{i} \in A_{r(y)}$, define
$$A^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}} : A_{r(x)} \to A_{r(x.\overleftarrow{\lambda} * \overrightarrow{i}.y)}$$
and
$$B^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}} : B_{\ell(y)} \to B_{\ell(x.\overleftarrow{\lambda} * \overrightarrow{i}.y)}$$
by
$$\overrightarrow{w} A^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}} = \overrightarrow{wh} \text{ and } \overleftarrow{w} B^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}} = \overleftarrow{kw}$$

where $h \in \overline{I}$, $k \in \overline{\Lambda}$ are such that $h\mathrm{R}x\lambda iy\mathrm{L}k \in S$. These maps are well defined. For, if $\overrightarrow{w_1} \in A_{r(x)}$ and $h_1 \in \overline{I}$ are such that $\overrightarrow{w} = \overrightarrow{w_1}$ and $\overrightarrow{h} = \overrightarrow{h_1}$ with $h\mathrm{R}x\lambda iy\mathrm{L}h_1$, then by Lemma 3.6, $w\mathrm{R}w_1 \Rightarrow wh = w_1h$. Since $h\mathrm{R}h_1 \Rightarrow w_1h\mathrm{R}w_2h$, $wh\mathrm{R}w_1h$, and hence $\overrightarrow{wh} = \overrightarrow{w_1h_1}$. We show that $\xi = \{A^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}}, B^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}}\}$ is an enrichment of (A, B) relative to $*$. Clearly (M_1) holds. To verify (M_2), take any
$$A^{x,y}_{\overleftarrow{\lambda}, \overrightarrow{i}} : A_{r(x)} \to A_{r(x.\overleftarrow{\lambda} * \overrightarrow{i}.y)}$$

with $x\mathrm{R}x.\overleftarrow{\lambda} * \overrightarrow{i}.y$, and let $h \in \overline{I}$ be such that $h\mathrm{R}x\lambda iy$. Then for any $\overrightarrow{w} \in A_{r(x)}$,

$$\overrightarrow{w}A^{x,y}_{\overleftarrow{\lambda},\overrightarrow{i}} = \overrightarrow{wh} = \overrightarrow{w},$$

since by Lemma 3.5, $w\mathrm{R}wh$. Hence $A^{x,y}_{\overleftarrow{\lambda},\overrightarrow{i}} = id$. Dually, we have $B^{x,y}_{\overleftarrow{\lambda},\overrightarrow{i}} = id$ whenever $y\mathrm{L}x.\overleftarrow{\lambda} * \overrightarrow{i}.y$.

Now let

$$W = W(U, A, B, *, \xi) = \{(\overleftarrow{\lambda}, x, \overrightarrow{i}) : x \in U;\ \overleftarrow{\lambda} \in A_{r(x)},\ \overrightarrow{i} \in B_{\ell(x)}\}$$

and define a multiplication on W by (4.1). Note that ξ satisfies $(M_3) - (M_5)$ if and only if the multiplication on W is associative. We verify $(M_3) - (M_5)$ by establishing the associativity of the multiplication. To this end, define $\gamma : S \to W$ by

$$s\gamma = (\overrightarrow{ss^*}, s^{00}, \overleftarrow{s^*s})$$

where $s^0 \in V_U(s)$ and $s^{00} \in V_U(s^0)$. Then γ is bijective map with inverse $\mu : W \to S$ given by $(\overleftarrow{\lambda}, x, \overrightarrow{i})\mu = \lambda xi$. Multiplication is preserved by γ, since

$$\begin{aligned}(s\gamma)(t\gamma) &= (\overrightarrow{ss^*}, s^{00}, \overleftarrow{s^*s})(\overrightarrow{tt^*}, t^{00}, \overleftarrow{t^*t}) \\ &= (\overrightarrow{ss^*h}, s^{00}.\overleftarrow{s^*s} * \overrightarrow{tt^*}.t^{00}, \overleftarrow{t^*}t) \\ &= (\overrightarrow{ss^*h}, (st)^{00}, \overleftarrow{kt^*t}) \\ &= (st)\gamma,\end{aligned}$$

the last step follows, since U is an orthodox transversal and $s^0 \in V_U(s)$, $s^{00} \in V_U(s^0)$, $t^0 \in V_U(t)$, $t^{00} \in V_U(t^0)$, $h \in \overline{I}$, $\overrightarrow{k} \in \overline{\Lambda}$ with $h\mathrm{R}s^{00}.(s^*stt^*)t^{00}\mathrm{L}k$. This implies that the multiplication in W is associative and γ is an isomorphism of regular semigroups. In particular ξ satisfies $(M_3) - (M_4)$ and hence ξ is an enrichment of (A, B) relative to $*$. Hence by the direct part of the theorem, $\overline{W} = \overline{W}(U; A, B; *, \xi)$ is a regular semigroup with an orthodox transversal $U = S^0SS^0. \cong S^0\eta.W.S^0\eta$, since $S \cong W$ and the proof of the theorem is complete.

Lemma 4.4 *The maps* $\{A^{x,y}_{a,b}, B^{x,y}_{a,b}\}$ *in the statement of the Theorem 4.3 are base point preserving maps if and only if* $S^0 = S^0\eta$ *is a quasi-ideal of* W.

Proof. Suppose that the maps $\{A^{x,y}_{a,b}, B^{x,y}_{a,b}\}$ are base point preserving maps.

Let $(r(x), x, \ell(x)), (r(y), y, \ell(y)) \in S^0$ and $(a, z, b) \in W$. Then

$$(r(x), x, \ell(x))(a, z, b)(r(y), y, \ell(y)) = (r(m), m, \ell(m)) \in S^0 (= S^0\eta)$$

where $m = x.\ell(x) * a.x.bB^{x,z}_{\ell(x),a} * r(y).y \in S$. So S^0 is a quasi-ideal of W.

Conversely, assume that S^0 is a quasi-ideal of W. Then

$$\begin{aligned}&(r(x)A^{x,y}_{a,b}, x.a * b.y, \ell(y)B^{x'y}_{a,b})\\ &\quad = (r(x), x, a)(b, y, \ell(y))\\ &\quad = (r(e), e, \ell(e))(r(x), x, a)(b, y, \ell(y))(r(f), f, \ell(f)) \text{ by } (M_2) \text{ and } (N_2)\\ &\quad \in S^0,\end{aligned}$$

where $e, f \in E(S^0)$ are such that $e\mathrm{R}x, f\mathrm{L}y$.

This implies

$$r(x)A^{x,y}_{a,b} = r(x.a * b.y) \text{ and } \ell(y)B^{x,y}_{a,b} = \ell(x.a * b.y).$$

Hence $A^{x,y}_{a,b}, B^{x,y}_{a,b}$ are base point preserving maps.

Note that when S^0 is a quasi-ideal orthodox transversal of S then $S^0SS^0 = S^0$. So S^0 is both an $S-$orthodox transversal and S^0-orthodox transversal. The following is the quasi-ideal version of the main theorem.

Theorem 4.5 *Let S^0 be an orthodox semigroup and let (A, B) be an S^0- pair. Let $*$ be a $B \times A$ matrix over S satisfying $(N_1), (N_2)$ and the following condition:*

$$\begin{aligned}(N_3)\ (i)\ & e.(b * aA(r(f), r(f')))f' = e.b * a.f'\\ (ii)\ & e'.(bB(\ell(e), \ell(e') * a)f = e'.b * a.f\end{aligned}$$

for all $e, e', f, f' \in E(S^0)$ with $\ell(e) \geq \ell(e'), r(f) \geq r(f'), a \in A_{r(f)}, b \in B_{\ell(e)}$. Then

$$W = W(S; A, B, *) = \{(a, x, b) : x \in S^0; a \in A_{r(x)}, b \in B_{\ell(x)}\}$$

is a regular semigroup under the multiplication

$$(a, x, b)(c, y, d) = (aA(r(x), r(z)), z, dB(\ell(y), \ell(z))$$

where $z = x.b * c.y$. The map $\eta : S \to W, x\eta = (r(x), x, \ell(x))$ is an injective *homomorphism of S to W. If the identity S with $S\eta$, via η, then S is a quasi-ideal orthodox transversal of W.*

Conversely, every regular semigroup with a quasi-ideal orthodox transversal can be constructed in this way.

Proof. For each quadruple (x, y, b, a), where $x, y \in S, b \in B_{\ell(x)}, a \in A_{r(y)}$, let

$$A^{x,y}_{b,a} = A(r(x), r(x.b * a.y)) \text{ and } B^{x,y}_{b,a} = B(\ell(y), \ell(x.b * a.y)).$$

Clearly the system $\xi = \xi(A, B) = \{A^{x,y}_{b,a}, B^{x,y}_{b,a}\}$ satisfies (M_1) and (M_2). Using (N_3) we get

$$\begin{aligned}(x.b * a.y)(c.B^{x,y}_{b,a} * d)z &= (x.b * a.y)(cB(\ell(y), \ell(x.b * a.y))z * d \\ &= x.b * a.y.c * d.z \\ &= x.b * aA(r(y), r(y.c * d.z))(y.c * d.z) \\ &= (x.b * aA^{y,z}_{c,d})(y.c * d.z),\end{aligned}$$

which implies $(M_3) - (M_5)$. Thus ξ is an enrichment of (A, B) relation to $*$. Then $W = (S^0, A, B; *) = W(S^0; A, B, *, \xi)$ and the direct part of the theorem follows from the direct part of Theorem 4.3 except perhaps the fact that $S(= S\eta)$ is a quasi-ideal of W. But this is immediate from Lemma 4.4, since $A^{x,y}_{b,a}, B^{x,y}_{b,a}$ are base point preserving maps.

Conversely, suppose S^0 is a quasi-ideal orthodox transversal of S. Let (A, B) be an S^0-pair with a $B\times A$ matrix over S^0, as in the converse part of Theorem 4.3. Then $*$ satisfies (N_1) and (N_2). We now show that $*$ also satisfies (N_3).

Take any $\overleftarrow{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)}$ and $r(f) \geq r(f')$. Then $\overrightarrow{i}A(r(f), r(f')) = \overrightarrow{if'}$ and,

$$\begin{aligned}e(\overleftarrow{\lambda} * \overrightarrow{i}A(r(f), r(f')))f' &= e(\overleftarrow{\lambda} * \overrightarrow{if'})f' \\ &= e((\lambda\beta)(if'\alpha))f' \\ &= e((\lambda\beta)(i\alpha))f' \text{ by Lemma 3.3.} \\ &= e(\overleftarrow{\lambda} * \overrightarrow{i})f'.\end{aligned}$$

Hence $(N_3)(i)$ is satisfied. A dual argument proves $(N_3)(ii)$. Hence by the direct part of the theorem, $W = W(S; A, B; *)$ is a regular semigroup containing $S(= S\eta)$ as a quasi-ideal orthodox transversal of W. Finally, as in the proof of Theorem 4.3 the map $\gamma : T \to W$ is an isomorphism of regular semigroups.

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