Regular Semigroups with a $S^0 -$ Orthodox Transversal

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Abstract

In this paper, we consider another generalization for quasi-ideal orthodox transversal, the so-called $S^0$-orthodox transversals. We give a structure theorem for regular semigroups with $S^0$-orthodox transversals. If $S^0$ is a $S^0$-orthodox transversal of $S$ then $S$ can be described in terms of $S^0$.

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1 Introduction

Let $S$ be a regular semigroup and $S^0$ be a regular subsemigroup of $S$. A natural question that has been considered by many authors is to what extent is $S$ determined by $S^0$? The concept of an inverse transversal is one of the answer to this question. Recall that an inverse transversal of a regular semigroup $S$ is an inverse subsemigroup $S^0$ that contains precisely one inverse for every $x \in S$. In 1982, Blyth and McFadden introduced the class of regular semigroups with an inverse transversal [1].

Recently, the concept of inverse transversal was generalized by many authors [2-10]. In particular, the concept of orthodox transversals was introduced by Chen Jianfei [2] as a generalization of inverse transversals. Chen Jianfei obtained an excellent structure theorem for regular semigroups with quasi-ideal orthodox transversals. In 2007, Xiangjun Kong [7] constructed regular semigroups with quasi-ideal orthodox transversals by a simpler format set. In 2009, Xiangjun Kong and Xianzhong Zhao [10] gave a structure theorem for regular semigroups with quasi-ideal orthodox transversals by two orthodox
semigroups. Hence the general case of orthodox transversals is to be considered. The main results are the sets

\[ I = \{ aa^0 : a \in S, a^0 \in V(a) \cap S^0 \} \]

and

\[ \Lambda = \{ a^0 a : a \in S, a^0 \in V(a) \cap S^0 \} \]

are two components of regular semigroups with orthodox transversals. Chen-Jianfei [2] have shown that \( I \) and \( \Lambda \) are subbands if \( S^0 \) is a quasi-ideal orthodox transversal of \( S \). Though each element of the sets \( I \) and \( \Lambda \) is an idempotent, they are necessarily subbands of \( S \). In 2001, Chen Jianfei and Guo Yugi [3] shown that, if \( S^0 \) is an orthodox transversal of \( S \), then the semi bands \( \langle I \rangle \) and \( \langle \Lambda \rangle \) generated by \( I \) and \( \Lambda \) respectively are bands. In this paper, we consider another generalization for quasi-ideal orthodox transversal, called \( S^0 \)-orthodox transversals. We give a structure theorem for regular semigroups with \( S^0 \)-orthodox transversals. This is also one of the answer to our question. That is, if \( S^0 \) is a \( S^0 \)-orthodox transversal of \( S \) then \( S \) can be described in terms of \( S^0 \).

Section 2 presents some necessary notation and known results. In section 3, we introduce two new subclasses, \( S^- \)-orthodox transversals and \( S^0^- \)-orthodox transversals of orthodox transversals and we obtain some basic properties of \( I \) and \( \Lambda \) when \( S \) is an \( S^0^- \)-orthodox transversal. In section 4, we give a structure theorem for regular semigroup with \( S^0^- \)-orthodox transversals. When \( S^0 \) is a quasi ideal of \( S \), our theorem simplifies considerably.

## 2 Preliminaries

We adopt the terminology, notation and results of [2] and [3].

**Definition 2.1** Let \( S \) be a semigroup and \( S^0 \) a subsemigroup of \( S \). We call \( S^0 \) an orthodox transversal of \( S \) if the following conditions are satisfied.

(i) \( V_{S^0}(x) \neq \phi \) for all \( x \in S \).

(ii) if \( x, y \in S \) and \( \{ x, y \} \cap S^0 \neq \phi \), then \( V_{S^0}(x)V_{S^0}(y) \subseteq V_{S^0}(yx) \).

Note that if \( S^0 \) is an orthodox transversal of \( S \), then \( S \) is regular by (i) and \( S^0 \) is an orthodox subsemigroup of \( S \) by (ii).

**Theorem 2.2** Let \( S \) be a regular semigroup and \( S^0 \) a quasi-ideal orthodox transversal of \( S \). Then
(i) \( I \cap \Lambda = E(S^0) \)
(ii) \( I = \{ e \in E(S) : (\exists e^* \in E(S^0)), eLe^* \} \)
\[ \Lambda = \{ f \in E(S) : (\exists f^+ \in E(S^0)), fRf^+ \} \]
(iii) \( IE(S^0) \subseteq I, E(S^0)\Lambda \subseteq \Lambda \).
(iv) \( I \) and \( \Lambda \) are subbands of \( S \).

**Theorem 2.3** Let \( S^0 \) be a quasi-ideal orthodox transversal of a regular semigroup \( S \). Then

(i) if \( e \in I \) (or \( \Lambda \)) then \( V_{S^0}(e) \subseteq E(S^0) \).
(ii) if \( x \in S \) and \( x^0 \in V_{S^0}(x) \), then \( V_{S^0}(x) = V_{S^0}(x^0)x^0V_{S^0}(xx^0) \).
(iii) if \( V_{S^0}(x) \cap V_{S^0}(y) \neq \phi \) for any \( x, y \in S \), then \( V_{S^0}(x) = V_{S^0}(y) \).

**Theorem 2.4** Let \( S^0 \) be an orthodox transversal of \( S \), then the Green relation \( \mathbb{H} \) on \( S \) saturates \( S^0 \). (That is, \( S^0 \) is a union of some \( \mathbb{H} \)-classes on \( S \).) In particular, the maximum idempotent-separating congruence on \( S \) saturates \( S^0 \).

**Theorem 2.5** Let \( S^0 \) be an orthodox transversal of \( S \). Then \( S \) is an orthodox semigroup if and only if for every \( a, b \in S \), \( V_{S^0}(a)V_{S^0}(b) \subseteq V_{S^0}(ba) \).

**Lemma 2.6** If \( S^0 \) is an orthodox transversal of \( S \) then for any \( a, b \in S^0 \), \( V(a) \cap V(b) \neq \phi \Rightarrow V_{S^0}(a) = V_{S^0}(b) \).

**Lemma 2.7** Let \( S^0 \) be an orthodox transversal of \( S \). For \( e \in S \), if \( V_{S^0}(e) \cap E(S^0) \neq \phi \), then \( V_{S^0}(e) \subseteq E(S^0) \).

**Theorem 2.8** Let \( S^0 \) be an orthodox transversal of \( S \). The semiband \( \langle I \rangle \) (respectively \( \langle \Lambda \rangle \)) generated by \( I \) (respectively \( \Lambda \)) is a subband of \( S \).

Note that if \( S^0 \) is an orthodox transversal of \( S \) then \( I \) is a band if and only if \( E(S^0)I \subseteq I \).

### 3 \( S^0 \)– ORTHODOX TRANSVERSALS

**Definition 3.1** Let \( S^0 \) be an orthodox transversal of \( S \). \( S^0 \) is said to be an \( S \)–orthodox transversal of \( S \) if \( I \) and \( \Lambda \) are subbands of \( S \).

**Definition 3.2** Let \( S^0 \) be an \( S \)–orthodox transversal of \( S \). Then \( S^0 \) is said to be an \( S^0 \)–orthodox transversal of \( S \) if the regular semigroup \( S^0SS^0 \) is an orthodox transversal of \( S \).
It is clear that a quasi-ideal orthodox transversal is an $S^0$—orthodox transversal. We denote $S^0SS^0$ by $U$.

**Lemma 3.3** Let $S^0$ be an $S^0$—orthodox transversal of $S$. If $i \in I$, then $iRe$ for some $e \in E(U)$ implies $i \in E(U)$; if $\lambda \in \Lambda$, then $\lambda Le$ for some $e \in E(U)$ implies $\lambda \in E(U)$.

**Proof.** If $iRe$ then $i = ei = eii^* \in S^0SS^0 = U$, and hence $i \in E(U)$. The second statement can be proved dually.

**Lemma 3.4** Let $S^0$ be an $S^0$—orthodox transversal of $S$. If $i \in I$ (or $\Lambda$) then $V_{S^0}(i) \subseteq E(U)$.

**Proof.** Let $i \in I$. Take $i^* \in E(S^0)$ such that $iLi^*$, and suppose that $x \in V_{S^0}(i)$ and $x^0 \in V_{S^0}(x)$. Since $x^0x \in E(S^0)$, $x^0xi \in S^0SS^0 = U$ and hence $x^0xi \in E(U)$. On the other hand, $i^*xx^0 \in E(U)$ since $U$ is orthodox. Therefore,

$$
x^0 = x^0xx^0 = x^0xiix^0 \text{ since } x \in V_{S^0}(i) = x^0xi.i^*xx^0 \text{ since } iLi^* = E(U).E(U) \subseteq E(U).
$$

Therefore, $x \in E(U)$, since $U$ is orthodox. Thus $V_{S^0}(i) \subseteq E(U)$.

Define

$$
\tilde{I} = \{i \in E(S) : (\exists i^* \in E(U)) : i^*Li\}
\tilde{\Lambda} = \{\lambda \in E(S) : (\exists \lambda^* \in E(U)) : \lambda'^*\Lambda\}
$$

Clearly $\tilde{I}$ and $\tilde{\Lambda}$ are subbands of $S$.

For each $e \in E(U)$, let

$$
I_e = \{i \in \tilde{I} : (\exists i^* \in E(U)) : i^*Re\}
\Lambda_e = \{\lambda \in \tilde{\Lambda} : (\exists \lambda^* \in E(U)) : \lambda'^*Le\}.
$$

**Lemma 3.5** Let $S^0$ be an $S^0$—orthodox transversal of $S$. Then $I_e$ and $\Lambda_e$ are rectangular bands.

**Proof.** Let $i, i_1 \in I_e$. Then there exist $i^*, i_1^* \in E(U)$ such that $i^*ReRi_1^*$. Since $S^0$ is $S^0$—orthodox transversal, $ii_1 \in I$ and $i^*i_1^* \in E(U)$. Further, since $i^*ReRi_1^*$, $i^*i_1^* = i_1^*Re$. Hence $I_e$ is a band. Let $i, i_1 \in I_e$. Then by Lemma 2.6, $V_{S^0}(ii_1i) = V_{S^0}(i)$. Since $E(U)$ is a band, so it is a semilattice of rectangular
bands. Therefore $ii_i i$ and $i$ are in the same rectangular band. Hence

$$ii' i = i.$$

Therefore $I_e$ is a rectangular band. Similarly, $\Lambda_e$ is also a rectangular band.

**Lemma 3.6** Let $S^0$ be an $S^0$-orthodox transversal of $S$. For any $i_1, i_2, i_3 \in I$ with $i_3 R i_1$, we have $i_3 i_2 = i_1 i_2$. Dually, for any $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, with $\lambda_3 L \lambda_1$, we have $\lambda_2 \lambda_3 = \lambda_2 \lambda_1$.

**Proof.** If $i_1, i_2, i_3 \in I$ then for some $i_1^*, i_2^*, i_3^* \in E(U)$, we have $i_1^* L i_1, i_2^* L i_2$ and $i_3^* L i_3$. If $i_3 R i_1$, then by Green’s lemma, $i_3 i_1^* = i_1$, and hence $i_3 i_2 = i_3 (i_1^* i_2) = (i_3 i_1^*) i_2 = i_1 i_2$. The second statement can be proved dually.

Maintaining the notation followed in [5], the following theorem is similar to Theorem of 2.5 of [5].

**Theorem 3.7** The association $r(e) \mapsto A_{r(e)}$, $(r(e), r(f)) \mapsto A(r(e), r(f))$ where

$$A_{r(e)} = I_e / R = \{ \overline{i} \in I / R : (\exists i^* \in E(U)) i^* R e \}$$

with $\overline{i} = r(e)$ as base point and where the map

$$A(r(e), r(f)) : A_{r(e)} \to A_{r(f)}$$

is given by $\overline{i} A(r(e), r(f)) = \overline{i e}$, defines a functor $A : E(U)/R \to P$.

Dually, the association, $\ell(e) \mapsto B_{\ell(e)}, (\ell(e), \ell(f)) \mapsto B(\ell(e), \ell(f))$, where $B_{\ell(e)} = \bar{\Lambda}_e / L = \{ \overline{\lambda} \in \bar{\Lambda} / L : (\exists \lambda L \in E(U)) \lambda L e \}$ with $\overline{\lambda} = \ell(e)$ as base point, and where the map $B(\ell(e), \ell(f)) : B_{\ell(e)} \to B_{\ell(f)}$ is given by $\overline{\lambda} B(\ell(e), \ell(f)) = \overline{f \lambda}$ defines a functor $B : E(U)/L \to P$.

## 4 Main Theorem

Let us define $S^0$-pair for an orthodox semigroup $S^0$.

**Definition 4.1** Let $S^0$ be an orthodox semigroup. By an $S^0$-pair $(A, B)$ we mean a pair of functors

$$A : E(S^0)/R \to P, B : E(S^0)/L \to P.$$

Given an $S^0$-pair $(A, B)$, a $B \times A$ matrix over $S^0$ is a function
\[ * : (b, a) \mapsto b * a : \bigcup_{\ell(e) \in E(S^0)/L} B_{\ell(e)} \times \bigcup_{r(f) \in E(S^0)/R} A_{r(f)} \to S^0 \]

**Definition 4.2** Let \((A, B)\) be an \(S^0\)-pair with a \(B \times A\) matrix \(*\) over \(S\). By an *enrichment* \(\xi = \xi(A, B)\) of \((A, B)\) relative to \(*\) we mean a family of maps

\[
A^{x,y}_{b,a} : A_{r(x)} \to A_{r(x,b*a,y)}, \quad B^{x,y}_{b,a} : B_{\ell(y)} \to B_{\ell(x,b*a,y)}
\]

where \(x, y \in S^0, b \in B_{\ell(x)}, a \in A_{r(y)}\), such that

1. \((M_1)\) \(A^{x,y}_{\ell(x),r(y)} = A(r(x), r(xy))\) and \(B^{x,y}_{\ell(x),r(y)} = B(\ell(y), \ell(xy))\),
2. \((M_2)\) if \(xRx.b * a.y\) then \(A^{x,y}_{b,a} = \text{id}\); and if \(yLx.b * a.y\) then \(B^{x,y}_{b,a} = \text{id}\),
3. \((M_3)\) \(A^{x,y}_{b,a} A^{x,y,z}_{c,b,a} = A^{x,y,c,d}_{b,a} A^{x,y}_{c,d}\),
4. \((M_4)\) \(B^{y,z}_{c,d} B^{x,y,c,d}_{b,a} = B^{x,y,c,d}_{c,B^{y,z}_{b,a}}\),
5. \((M_5)\) \(x.b * a.y.c B^{x,y}_{b,a} * d.z = x.b * a A^{y,z}_{c,d} y.c * d.z\)

for all \(x, y, z \in S, b \in B_{\ell(x)}, a \in A_{r(y)}, c \in B_{\ell(y)}, d \in A_{r(z)}\).

**Theorem 4.3** Let \(S^0\) be an orthodox semigroup and let \((A, B)\) be an \(S^0\)-pair. Let \(*\) be a \(B \times A\) matrix over \(S^0\) satisfying

1. \((N_1)\) if \(b \in B_{\ell(e)}\) and \(a \in A_{r(f)}\) then \(b * a \in \ell(e) . S r(f)\).
2. \((N_2)\) for any \(b \in B_{\ell(e)}, a \in A_{r(f)}, b * r(f), \ell(e) * a \in \ell(e)r(f)\).

Let \(\xi\) be an enrichment of \((A, B)\) relative to \(*\). Then the set

\[ W = W(S^0; A, B; *; \xi) = \{(a, x, b) : x \in S^0, a \in A_{r(x)}, b \in B_{\ell(x)}\} \]

is a regular semigroup under the multiplication

\[ (a, x, b)(c, y, d) = \{aA^{x,y}_{b,c}, x.b*c,y, dB^{x,y}_{b,c}\} \quad (4.1) \]

The map \(\eta : S^0 \to W, x\eta = (r(x), x \ell(x))\) is an injective homomorphism of \(S^0\) to \(W\). If we identify \(S^0\) with \(S^0\eta\), via \(\eta\), then \(S^0\) is an \(S^0\)-orthodox transversal of \(S\).

Conversely, every regular semigroup with an \(S^0\)-orthodox transversal can be constructed in this way.
Proof. The associativity of the multiplication follows from \((M_3)\) – \((M_5)\). We first prove that \(\eta\) is an injective homomorphism. Clearly \(\eta\) is one-to-one. Since \(A_{\ell(x),\ell(y)}^{x,y}, B_{\ell(x),\ell(y)}^{x,y}\) are base point preserving function by \((M_1)\), we get

\[
\begin{align*}
x\eta.y\eta &= (r(x), x, \ell(x)) (r(y), y, \ell(y)) \\
&= (r(xy), xy, \ell(xy)) \\
&= (xy)\eta.
\end{align*}
\]

Hence \(\eta\) is an injective homomorphism.

Let \((a, x, b) \in W\). Then \(x \in S^0\), let \(x^* \in V_{S^0(x)}\), by \((N_2)\) and \((M_2)\),

\[
(a, x, b) (r(x^*), x^*, \ell(x^*)) (a, x, b) = (a, xx^*, \ell(x^*)) (a, x, b) = (a, x, b)
\]

and

\[
\begin{align*}
r(x^*), x^*, \ell(x^*)) (a, x, b) (r(x^*), x^*, \ell(x^*)) &= (r(x^*), x^*, \ell(x^*)) (r(x^*), x^*, \ell(x^*)) \\
&= (r(x^*), x^*, \ell(x^*))
\end{align*}
\]

so that \((r(x^*), x^*, \ell(x^*)) \in V_{S^0}((a, x, b))\), since we can identify \(S^0\) with \(S^0 \eta\), \(V_{S^0}((a, x, b)) \neq \emptyset\).

Moreover, for any \((a, x, b) \in W\),

\[
V_{S^0}((a, x, b)) = \{(r(x^*), x^*, \ell(x^*)) : x^* \in V_{S^0}(x)\}.
\]

Hence \(W\) is a regular semigroup. Now let \((a, x, b) \in W\) and \((r(y), y, \ell(y)) \in S^0 \eta \cong S^0\). Let \((r(x^*), x^*, \ell(x^*)) \in V_{S^0}((a, x, b))\) and \((r(y^*), y^*, \ell(y^*)) \in V_{S^0}((r(y), y, \ell(y)))\). Then

\[
(r(x^*), x^*, \ell(x^*)) (r(y^*), y^*, \ell(y^*)) \in V_{S^0}((a, x, b)) V_{S^0}((r(y), y, \ell(y)))
\]

\[
\Rightarrow (r(x^* y^*), x^* y^*, \ell(x^* y^*)) \in V_{S^0}((a, x, b)) V_{S^0}((r(y), y, \ell(y)))
\]

Consider \((r(y), y, \ell(y)) (a, x, b) = (r(yx), yx, bB_{\ell(y),a}^{x,y})\) by \((N_2)\) and \((M_2)\). Next we prove that \((r(x^* y^*), x^* y^*, \ell(x^* y^*)) \in V_{S^0}((r(yx), yx, bB_{\ell(y),a}^{x,y}))\). But this is immediately follows, since \(S^0\) is an orthodox semigroup and by \((M_2)\) and \((N_2)\).

Therefore, for any \((a, x, b) \in W\), \((r(y), y, \ell(y)) \in S^0 \eta \cong S^0\),

\[
V_{S^0}((a, x, b)) V_{S^0}((r(y), y, \ell(y))) \subseteq V_{S^0}((r(y), y, \ell(y))(a, x, b))
\]

Hence \(S^0\) is an orthodox transversal.
Note that by \((M_2)\),

\[
E(W) = \{(a, x, b) \in W : x.(b*a).x = x\}.
\]

Consider the sets

\[
I = \{(a, x, b) \in E(W) : (\exists((r(x_1), x_1, \ell(x_1))) \in E(S^0) (a, x, b)L(r(x_1), x_1, \ell(x_1))}\}
\]

\[
\Lambda = \{(c, y, d) \in E(W) : (\exists((r(y_1), y_1, \ell(y_1))) \in E(S^0) (c, y, d)R(r(y_1), y_1, \ell(y_1))}\}.
\]

Let \((a, x, b), (c, y, d) \in I\). Take \((r(x_1), x_1, \ell(x_1)) \in E(S^0)\) such that \((r(x_1), x_1, \ell(x_1)) \in L(a, x, b)\), then

\[
(a, x, b)(c, y, d) = (a, x, b)(r(x_1), x_1, \ell(x_1))(c, y, d)
\]

But \((r(x_1), x_1, \ell(x_1)) \in E(S^0)\) by \((N_2)\). Thus

\[
(a, x, b)(c, y, d) = (a, x, b)(r(x_1), x_1, \ell(x_1)) \subseteq IE(S^0) \subseteq I,
\]

by Theorem 2.2(iii). Hence \(I\) is a band. Similarly, we can prove \(\Lambda\) is a band. So \(S^0\) is a \(S\)–orthodox transversal of \(S\). Since \(S^0\) is an orthodox transversal of \(S\), \(S^0WS^0\) is a regular subsemigroup of \(S\). Let \((a, x, b) \in W, (r(x_1), x_1, \ell(x_1)), (r(x_2), x_2, \ell(x_2)) \in S^0WS^0\), then

\[
(r(x_1), x_1, \ell(x_1))(a, x, b)(r(x_2), x_2, \ell(x_2)) = (r(m), m, \ell(m)) \text{ by \((N_2)\)}
\]

where \(m = x_1.\ell(x_1) * a.x.b * r(x_2).x_2 \in S^0WS^0\).

Therefore,

\[
U = S^0WS^0 = S^0\eta.W.S^0\eta
\]

\[
= \{(r(m), m, \ell(m)) : m \in S^0WS^0\}.
\]

By \((N_2)\), \(U\) is an orthodox transversal of \(W\). Let \((A, B)\) be an \(U\)–pair with a \(B \times A\) matrix \(*\) over \(U\) and \(\xi = \xi(A, B)\) be an enrichment of \(A, B\) relative to \(*\).

Then the set

\[
\mathcal{W} = \mathcal{W}(U; A, B, *, \xi) = \{((a, x, b) : x \in U, a \in A_{r(x)}, b \in B_{\ell(x)}\}
\]

is a regular semigroup under the multiplication given by (4.1) and \(U\) is an
orthodox transversal of $\mathbb{W}$.

Conversely, suppose that $S^0$ is an $S^0$—orthodox transversal of $S$. Let $(A, B)$ be the $S^0SS^0$—pair defined in Theorem 3.7, we define a $B \times A$ matrix $\lambda$ over $S^0SS^0 = U$ as follows. Fix an $R$— invariant map $\alpha : \mathbb{T} \rightarrow \mathbb{T}$ so that $\alpha$ is constant on each $R$— class of $\mathbb{T}$. Similarly fix an $L$— invariant map $\beta : \overline{\Lambda} \rightarrow \overline{\Lambda}$ so that $\beta$ is constant on each $L$— class of $\overline{\Lambda}$. For each $\overline{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)},$

define

$$\overline{\lambda} * \overrightarrow{i} = (\lambda \beta)(i \alpha).$$

Clearly $\lambda$ is well defined. For, if $\overline{\lambda}_1 = \overline{\lambda}_2$, $\overrightarrow{i}_1 = \overrightarrow{i}_2$ then $\lambda_1 \beta = \lambda_2 \beta$, $i_1 \alpha = i_2 \alpha$ and so $((\lambda \beta)(i \alpha)) = ((\lambda_2 \beta)(i_2 \alpha))$. We show that $\lambda$ satisfies $(N_1)$ and $(N_2)$.

$(N_1)$ If $\overline{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)},$

then

$$\overline{\lambda} * \overrightarrow{i} = (\lambda \beta)(i \alpha)$$

easily $(e \lambda \beta)(i \alpha) f \in \ell(e)U r(f),$

$(N_2)$ If $\overline{\lambda} \in B_{\ell(e)}$, $\overrightarrow{i} \in A_{r(f)}$ then

$$\overline{\lambda} * r(f) = (\lambda \beta)(f \alpha)$$

since $(\lambda \beta) \in U = S^0SS^0$. Similarly, $\ell(e) * \overrightarrow{i} \in \ell(e)r(f)$.

For each quadruple $(x, y, \overrightarrow{i}, \overline{\lambda})$, where $x, y \in U$, $\overline{\lambda} \in B_{\ell(x)}$, $\overrightarrow{i} \in A_{r(y)}$, define

$$A^{x,y}_{\lambda, \overrightarrow{i}} : A_{r(x)} \rightarrow A_{r(x, \overrightarrow{i} \cdot y)}$$

and

$$B^{x,y}_{\lambda, \overrightarrow{i}} : B_{\ell(y)} \rightarrow B_{\ell(x, \overrightarrow{i} \cdot y)}$$

by

$$\overrightarrow{w} A^{x,y}_{\lambda, \overrightarrow{i}} = \overrightarrow{w} h \text{ and } \overrightarrow{w} B^{x,y}_{\lambda, \overrightarrow{i}} = \overrightarrow{k w}$$

where $h \in \mathbb{T}$, $k \in \overline{\Lambda}$ are such that $hRx \lambda iy Lk \in S$. These maps are well defined. For, if $\overrightarrow{w}_1 \in A_{r(x)}$ and $h_1 \in \mathbb{T}$ are such that $\overrightarrow{w} = \overrightarrow{w}_1$ and $\overrightarrow{h} = \overrightarrow{h}_1$ with $hRx \lambda iy Lh_1$, then by Lemma 3.6, $wRw_1 \Rightarrow wh = w_1h$. Since $hRh_1 \Rightarrow w_1hRw_2h$, $whRw_1h$, and hence $\overrightarrow{w} h = w_1h$. We show that $\xi = \{A^{x,y}_{\lambda, \overrightarrow{i}}, B^{x,y}_{\lambda, \overrightarrow{i}}\}$ is an enrichment of $(A, B)$ relative to $\lambda$. Clearly $(M_1)$ holds. To verify $(M_2)$, take any

$$A^{x,y}_{\lambda, \overrightarrow{i}} : A_{r(x)} \rightarrow A_{r(x, \overrightarrow{i} \cdot y)}$$
with \( xRx, \overrightarrow{\lambda} \ast i, y \), and let \( h \in T \) be such that \( hRx \lambda iy \). Then for any \( \overline{w} \in A_r(x) \),

\[
\overline{w} A_{\overrightarrow{\lambda}, i}^{x,y} = \overline{w} \overrightarrow{h} = \overline{w},
\]

since by Lemma 3.5, \( wRwh \). Hence \( A_{\overrightarrow{\lambda}, i}^{x,y} = id \). Dually, we have \( B_{\overrightarrow{\lambda}, i}^{x,y} = id \) whenever \( yLx. \overrightarrow{\lambda} \ast i, y \).

Now let

\[
W = W(U, A, B, \ast, \xi) = \{ (\overrightarrow{\lambda}, x, \overrightarrow{i}) : x \in U; \overrightarrow{\lambda} \in A_r(x), \overrightarrow{i} \in B_{\ell(x)} \}
\]

and define a multiplication on \( W \) by (4.1). Note that \( \xi \) satisfies \( (M_3) - (M_5) \) if and only if the multiplication on \( W \) is associative. We verify \( (M_3) - (M_5) \) by establishing the associativity of the multiplication. To this end, define \( \gamma : S \rightarrow W \) by

\[
s\gamma = (ss^s, s^{00}, s^*s)
\]

where \( s^0 \in V_U(s) \) and \( s^{00} \in V_U(s^0) \). Then \( \gamma \) is bijective map with inverse \( \mu : W \rightarrow S \) given by \( (\overrightarrow{\lambda}, x, \overrightarrow{i})\mu = \lambda xi \). Multiplication is preserved by \( \gamma \), since

\[
(s\gamma)(t\gamma) = (ss^s, s^{00}, s^*s)(tt^t, t^{00}, t^*t) = (ss^sh, s^{00}s^*s \ast tt^t, t^{00}, t^*t) = (ss^sh, (st)^{00}, kt^*t) = (st)\gamma,
\]

the last step follows, since \( U \) is an orthodox transversal and \( s^0 \in V_U(s) \), \( s^{00} \in V_U(s^0) \), \( t^0 \in V_U(t) \), \( t^{00} \in V_U(t^0) \), \( h \in T \), \( k \in \Lambda \) with \( hRs^{00}.(s^*s \ast t^*)t^{00}Lk \). This implies that the multiplication in \( W \) is associative and \( \gamma \) is an isomorphism of regular semigroups. In particular \( \xi \) satisfies \( (M_3) - (M_4) \) and hence \( \xi \) is an enrichment of \( (A, B) \) relative to \( \ast \). Hence by the direct part of the theorem, \( \overline{W} = \overline{W}(U; A, B; \ast, \xi) \) is a regular semigroup with an orthodox transversal \( U = S^0SS^0 \simeq S^0\eta.W.S^0\eta \), since \( S \simeq W \) and the proof of the theorem is complete.

**Lemma 4.4** The maps \( \{ A_{a,b}^{x,y}, B_{a,b}^{x,y} \} \) in the statement of the Theorem 4.3 are base point preserving maps if and only if \( S^0 = S^0\eta \) is a quasi-ideal of \( W \).

**Proof.** Suppose that the maps \( \{ A_{a,b}^{x,y}, B_{a,b}^{x,y} \} \) are base point preserving maps.
Let \((r(x), x, \ell(x)), (r(y), y, \ell(y)) \in S^0\) and \((a, z, b) \in W\). Then
\[
(r(x), x, \ell(x))(a, z, b)(r(y), y, \ell(y)) = (r(m), m, \ell(m)) \in S^0(= S^0\eta)
\]
where \(m = x.\ell(x) * a.x.bE_{\ell(x), a} * r(y).y \in S\). So \(S^0\) is a quasi-ideal of \(W\).

Conversely, assume that \(S^0\) is a quasi-ideal of \(W\). Then
\[
(r(x)A_{a,b}^{x,y}, x.a * b.y, \ell(y)B_{a,b}^{x,y}) = (r(x), x, a)(b, y, \ell(y))(r(f), f, \ell(f)) \text{ by } (M_2) \text{ and } (N_2)
\]
where \(e, f \in E(S^0)\) are such that \(eRx, fLy\).

This implies
\[
r(x)A_{a,b}^{x,y} = r(x.a * b.y) \text{ and } \ell(y)B_{a,b}^{x,y} = \ell(x.a * b.y).
\]

Hence \(A_{a,b}^{x,y}, B_{a,b}^{x,y}\) are base point preserving maps.

Note that when \(S^0\) is a quasi-ideal orthodox transversal of \(S\) then \(S^0S^0 = S^0\). So \(S^0\) is both an \(S\)–orthodox transversal and \(S^0\)–orthodox transversal. The following is the quasi-ideal version of the main theorem.

**Theorem 4.5** Let \(S^0\) be an orthodox semigroup and let \((A, B)\) be an \(S^0\)–pair. Let \(*\) be a \(B \times A\) matrix over \(S\) satisfying \((N_1), (N_2)\) and the following condition:

\[
(N_3) \quad (i) e.(b * aA(r(f), r(f')))f' = e.b * a.f' \\
(ii) e'.(bB(\ell(e), \ell(e') * a)f = e'.b * a.f
\]

for all \(e, e', f, f' \in E(S^0)\) with \(\ell(e) \geq \ell(e'), r(f) \geq r(f'), a \in A_r(f), b \in B_{\ell(e)}\). Then
\[
W = W(S; A, B, *) = \{(a, x, b) : x \in S^0; a \in A_r(x), b \in B_{\ell(x)}\}
\]
is a regular semigroup under the multiplication
\[
(a, x, b)(c, y, d) = (aA(r(x), r(z)), z, dB(\ell(y), \ell(z))
\]
where \(z = x.b * c.y\). The map \(\eta : S \to W, x\eta = (r(x), x, \ell(x))\) is an injective
homomorphism of $S$ to $W$. If the identity $S$ with $S_\eta$, via $\eta$, then $S$ is a quasi-ideal orthodox transversal of $W$.

Conversely, every regular semigroup with a quasi-ideal orthodox transversal can be constructed in this way.

**Proof.** For each quadruple $(x, y, b, a)$, where $x, y \in S, b \in B_{\ell(x)}, a \in A_{r(y)}$, let

$$A_{b,a}^{x,y} = A(r(x), r(x.b*a.y)) \text{ and } B_{b,a}^{x,y} = B(\ell(y), \ell(x.b*a.y)).$$

Clearly the system $\xi = \xi(A, B) = \{A_{b,a}^{x,y}, B_{b,a}^{x,y}\}$ satisfies ($M_1$) and ($M_2$). Using ($N_3$) we get

$$(x.b*a.y)(c.B_{b,a}^{x,y} * d)z = (x.b*a.y)(cB(\ell(y), \ell(x.b*a.y)))z * d$$

$$= x.b*a.y.c * d.z$$

$$= x.b*aA(r(y), r(y.c * d.z))(y.c * d.z)$$

$$= (x.b*aA_{c,d}^{y,z})(y.c * d.z),$$

which implies ($M_3$) - ($M_5$). Thus $\xi$ is an enrichment of $(A, B)$ relation to $\ast$. Then $W = (S^0, A, B; \ast) = W(S^0; A, B, \ast, \xi)$ and the direct part of the theorem follows from the direct part of Theorem 4.3 except perhaps the fact that $S(= S_\eta)$ is a quasi-ideal of $W$. But this is immediate from Lemma 4.4, since $A_{b,a}^{x,y}, B_{b,a}^{x,y}$ are base point preserving maps.

Conversely, suppose $S^0$ is a quasi-ideal orthodox transversal of $S$. Let $(A, B)$ be an $S^0$–pair with a $B \times A$ matrix over $S^0$, as in the converse part of Theorem 4.3. Then $\ast$ satisfies ($N_1$) and ($N_2$). We now show that $\ast$ also satisfies ($N_3$).

Take any $\lambda \in B_{\ell(e)}$, $i \in A_{r(f)}$ and $r(f) \geq r(f')$. Then $\lambda^i A(r(f), r(f')) = \bar{if}^f$ and,

$$e(\lambda^i A(r(f), r(f'))f') = e((\lambda^i \bar{if})f')f'$$

$$= e((\lambda \bar{i}(\alpha)f')f'$$

$$= e((\lambda \bar{i}(\alpha)f')f' \text{ by Lemma 3.3.}$$

$$= e(\lambda \bar{i} f')f'.$$

Hence ($N_3$)(i) is satisfied. A dual argument proves ($N_3$)(ii). Hence by the direct part of the theorem, $W = W(S; A, B; \ast)$ is a regular semigroup containing $S(= S_\eta)$ as a quasi-ideal orthodox transversal of $W$. Finally, as in the proof of Theorem 4.3 the map $\gamma : T \to W$ is an isomorphism of regular semigroups.
References


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