International Mathematical Forum, Vol. 7, 2012, no. 31, 1547 - 1552

Regular Semigroups with Inverse Transversals

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Abstract

In this paper, a structure theorem is obtained by a permissible double (R, Λ) which is about regular semigroup with inverse transversals. It improves a structure theorem which is obtained by McAlister and McFadden about regular semigroup with inverse transversals. Furthermore, we give some properties of regular semigroups with inverse transversals.

Keywords: regular semigroups; inverse transversals; isomorphism

1 Introduction and preliminaries

In 1982, Blyth and McFadden introduced regular semigroups with inverse transversals in [4], this type of semigroup has attracted much attention. An inverse subsemigroup S^0 of a regular semigroup S is an *inverse transversal* if $|V(x) \bigcap S^0| = 1$ for any $x \in S$, where $V(x)$ denotes the set of inverses of x. In this case, the unique element of $V(x) \bigcap S^0$ is denoted by x^0 and $(x^0)^0$ is denoted by x^{00}. An inverse transversal S^0 of a regular semigroup S is a *Q-inverse transversal* if $S^0SS^0 \subseteq S^0$. Let S be an regular semigroup with an inverse transversal S^0, and let

$$R(S) = \{x \in S \mid x^0x = x^0x^{00}\}, L(S) = \{a \in S \mid aa^0 = a^{00}a^0\},$$

$$I(S) = \{e \in E(S) \mid ee^0 = e\}, \Lambda(S) = \{f \in E(S) \mid f^0f = f\},$$

where $E(S) = \{x \in S \mid x^2 = x\}$ which is the idempotents of S. A band B is *left [right] regular* if $efe = ef$ $[efe = fe]$ for any $e, f \in B$. A *left [right] inverse semigroup* is an orthodox semigroup whose band of idempotents is left [right] regular. An inverse subsemigroup S^0 of a regular semigroup S is called a *S-inverse transversal* if $I(S)$ and $\Lambda(S)$ are subsemigroups of S. In [1] Tatsuhiko

Saito first show that $R(S)$ $[L(S)]$ is a subsemigroup of S if and only if $I(S)$ $[\Lambda(S)]$ is subsemigroup of S. In [5], Tang Xilin shows that $I(S)$ $[\Lambda(S)]$ is left [right] regular band with an inverse transversal $E(S^0)$. This means that, in the terminology of [2], every inverse transversal of S is a S-inverse transversal.

For convenience, $R(S)$ is denoted by R and $L(S)$ is denoted by L, and $I(S)$ is denoted by I and $\Lambda(S)$ is denoted by Λ, and the semilattice $E(S^0)$ is denoted by E^0. We list several known results which will be frequently used in this paper without special reference.

Let S be an regular semigroup with an inverse transversal S^0. In [6], we know that

$$L \cap R = S^0,\ E(R) = I,\ E(L) = \Lambda,\ I \cap \Lambda = E^0.$$

If S is a left [right] inverse semigroup, then

$$R = S\ [L = S],\ I = E(S)\ [\Lambda = E(S)].$$

If $g^0 \in E^0$, then $g^0 e = g^0 e^0$ for each $e \in I$ and $fg^0 = f^0 g^0$ for each $f \in \Lambda$. The regular subsemigroup $\langle E(S)\rangle$ generated by the idempotents of S is denoted by C, and we denote that $\langle\langle e\rangle\rangle = \langle E(eCe)\rangle$ for any e in $E(S)$. Clearly, eCe is regular, so is $\langle\langle e\rangle\rangle$. By [6], we know that $C^0 = C \cap S^0$ is an inverse transversal of C and

$$E^0 = E(C^0), E(C) = E(S),$$

$$I_C = \{e \in C \mid ee^0 = e\} = \{e \in E(S) \mid ee^0 = e\} = I,$$

and

$$\Lambda_C = \{f \in C \mid f^0 f = f\} = \{f \in E(S) \mid f^0 f = f\} = \Lambda.$$

Let S be a regular semigroup with an inverse transversal S^0, for any $a \in S$, define $\lambda_a \in \mathcal{P}_S$ (the semigroup of partial mappings of S) as follows:

$$\lambda_a : \langle\langle aa^0\rangle\rangle \to \langle\langle a^0 a\rangle\rangle,\ x \mapsto a^0 x a.$$

The composition of λ_a and λ_b in $\mathcal{P}_S$ is denoted by $\lambda_a\lambda_b$ for any $a, b \in S$.

Lemma 1.1[6] Let S be an regular semigroup with a Q-inverse transversal S^0, then R and L are orthodox semigroups.

Lemma 1.2[3] Let S be an regular semigroup with an inverse transversal S^0, and $e = aa^0, f = a^0 a$ for each $a \in S$. Then λ_a is an isomorphism, its inverse is $\lambda_a^{-1} : \langle\langle f\rangle\rangle \to \langle\langle e\rangle\rangle, y \mapsto aya^0$.

In a regular semigroup S with an inverse transversal S^0, the subsets I, Λ, R, L play important roles in studying the nature of this sort of semigroup. In this paper, a structure theorem is obtained by a permissible double (R, Λ) which is about regular semigroup with inverse transversals. It improves a structure theorem which is obtained by McAlister and McFadden about regular semigroup with inverse transversals. Furthermore, we give some properties of regular semigroups with inverse transversals.

2 The main results

Theorem 2.1 Let R and L be orthodox semigroups with a common Q-inverse transversal S^0. Suppose that Λ is a right regular band with an inverse transversal E^0. Let $\Lambda \times R \to S^0$ described by $(e, x) \mapsto e * x$ such that, for any $x, y \in R$ and for any $e, f \in \Lambda$,

(1) $(e * x)y = e * xy$ and $f(e * x) = fe * x$;

(2) $e * x = ex$ if $x \in E^0$ or $e \in E^0$.

Define a multiplication on the set $R| \times |\Lambda = \{(x, e) \in R \times \Lambda \mid x^0x = e^0\}$ by $(x, e)(y, f) = (x(e * y), [(e * y)^0(e * y)]f)$. Then $R| \times |\Lambda$ is a regular semigroup with a Q-inverse transversal which is isomorphic to S^0.

Conversely, every regular semigroup with a Q-inverse transversal can be constructed in this way.

Proof. We give an outline of the proof.

By using (1) and (2), we can calculate: for any $(x, e), (y, f), (z, g) \in R|\times|\Lambda$,

$$[x(e * y)]^0x(e * y) = (e * y)^0(e * y) = \{[(e * y)^0(e * y)]f\}^0,$$

$$[(x, e)(y, f)](z, g) = (x(e*y)(f*z), \{[(e*y)(f*z)]^0(e*y)(f*z)\}g) = (x, e)[(y, f)(z, g)]$$

and if $r, s \in S^0$, then $(r, r^0r)(s, s^0s) = (rs, (rs)^0rs)$.

Let $(x, e) \in R| \times |\Lambda$, then we have

$$x(e * x^0)x = x(e * x^0x) = x(e * e^0) = x(ee^0) = xe^0 = xx^0x = x$$

and

$$(e*x^0)x(e*x^0) = (e*x^0x)(e*x^0) = (ex^0x)(e*x^0) = ee^0(e*x^0) = e^0(e*x^0) = e^0e*x^0 = e*x^0.$$

Thus $e * x^0 = x^0$. By using this fact, we can prove that $(x^0, x^{00}x^0)$ is an inverse in the set $S^0| \times |E^0 = \{(r, r^0r) \mid r \in S^0\}$ of (x, e).

Thus $R| \times |\Lambda$ is a regular semigroup containing an inverse subsemigroup $S^0|\times|E^0$ which is isomorphic to S^0, and each element of $R|\times|\Lambda$ has an inverse in $S^0| \times |E^0$.

Let (r, r^0r) be an inverse in $S^0| \times |E^0$ of $(x, e) \in R| \times |\Lambda$. Then we have

$$(x, e) = (x, e)(r, r^0r)(x, e) = (x(e * r)x, \{[(e * r)x]^0(e * r)x\}e)$$

and

$$(r, r^0r) = (r, r^0r)(x, e)(r, r^0r) = (rx(e * r), [x(e * r)]^0 rx(e * r)).$$

Thus we have $x = x(e * r)x$ and $r = rx(e * r)$.
Since

$$e * r = e * rx(e * r) = (e * r)x(e * r),$$

thus $e * r = x^0$.
Since

$$x^0r^0 = (e * r)r^0 = e * rr^0 = e(rr^0),$$

thus x^0r^0 ia an idempotent in S^0, and so

$$rx = (rx)^{00} = (x^0r^0)^0 = x^0r^0.$$

Thus we have

$$x^0 = e * r = (e * r)r^0r = x^0r^0r = rxr = rxrx(e * r) = rx(e * r) = r.$$

Thus $S^0| \times |E^0$ is a Q-inverse transversal of $R| \times |\Lambda$.

Conversely, suppose that S is a regular semigroup with a Q-inverse transversal S^0. Let $\Lambda \times R \to S^0$ be a mapping given by $(e, x) \mapsto e * x = ex$. Then the mapping satisfied (1) and (2), and we can constructed a regular semigroup $R| \times |\Lambda = \{(x, e) \in R \times \Lambda \mid x^0x = e^0\}$ under a multiplication $(x, e)(y, f) = (xey, (ey)^0 eyf)$. By defining a mapping $R| \times |\Lambda \to S$ given by $(x, e) \mapsto xe$, we can prove that $R| \times |\Lambda \simeq S$.

Theorem 2.2 Let S be a regular semigroup with an inverse transversal S^0, then $\lambda_{a^0} = \lambda_a^{-1}$ for any $a \in S^0$.

Proof. For each $a \in S$, by Lemma 1.2, we know

$$\lambda_{a^0} : \langle\langle a^0a^{00}\rangle\rangle \to \langle\langle a^{00}a^0\rangle\rangle,\ x \mapsto a^{00}xa^0.$$
$$\lambda_a^{-1} : \langle\langle a^0a\rangle\rangle \to \langle\langle aa^0\rangle\rangle,\ y \mapsto aya^0.$$

Since

$$R(S) = \{x \in S \mid x^0x = x^0x^{00}\},\ L(S) = \{a \in S \mid aa^0 = a^{00}a^0\},\ L(S) \cap R(S) = S^0,$$

thus $aa^0 = a^{00}a^0, a^0a = a^0a^{00}$ for any $a \in S^0$.

It is obvious that $\mathrm{dom}\lambda_{a^0} = \langle\langle a^0a^{00}\rangle\rangle = \langle\langle a^0a\rangle\rangle = \mathrm{dom}\lambda_a^{-1}$, $\mathrm{ran}\lambda_{a^0} = \langle\langle a^{00}a^0\rangle\rangle = \langle\langle aa^0\rangle\rangle = \mathrm{ran}\lambda_a^{-1}$.

Suppose that $x \in \langle\langle a^0a^{00}\rangle\rangle$, then we have $x = x_1 \cdots x_n$, where $x_1, \cdots, x_n \in E(a^0a^{00}Ca^0a^{00})$, and $x_i = a^0a^{00}x_ia^0a^{00}$ for any $i \in \{1, \cdots, n\}$,

$$\begin{aligned} x\lambda_{a^0} &= a^{00}a^0a^{00}x_1a^0a^{00} \cdots a^0a^{00}x_na^0a^{00}a^0 \\ &= aa^0a^{00}x_1a^0a \cdots a^0ax_na^0 \\ &= aa^0ax_1a^0a \cdots a^0ax_na^0aa^0 \\ &= ax_1a^0a \cdots a^0ax_na^0, \end{aligned}$$

then $x = a^0ax_1a^0a\cdots a^0ax_na^0a$, thus $x \in \langle\langle a^0a\rangle\rangle = \langle\langle a^0a^{00}\rangle\rangle$, and $x_i = a^0a^{00}x_ia^0a^{00} = a^0ax_ia^0a$ for any $i \in \{1,\cdots,n\}$, where $x_1,\cdots,x_n \in E(a^0aCa^0a)$,

$$x\lambda_a^{-1} = aa^0ax_1a^0a\cdots a^0ax_na^0aa^0 = ax_1a^0a\cdots a^0ax_na^0,$$

thus $x\lambda_{a^0} = x\lambda_a^{-1}$ for any $a \in S^0$.

Let S be a regular semigroup with an inverse transversal S^0, and let $T = \bigcup_{a\in S}\lambda_a,\ e = aa^0,\ f = a^0a,\ g = bb^0,\ h = b^0b$ for any $a, b \in S$. As the same way in [3], we can define a multiplication on the set T by $\lambda_a \circ \lambda_b = \lambda_{ab}$, and λ_{ab} is an isomorphism from $\langle\langle ab(ab)^0\rangle\rangle$ onto $\langle\langle (ab)^0ab\rangle\rangle$, then we have the following result:

Theorem 2.3 Let S be a regular semigroup with an inverse transversal S^0, then T is a regular semigroup with a multiplication by $\lambda_a \circ \lambda_b = \lambda_{ab}$.

Proof. The operation is well-defined:

For any $\lambda_a, \lambda_b \in T$, it is obvious that $\lambda_{ab} \in T$, and $x\lambda_{ab} = (ab)^0x(ab)$.

The operation is associative:

$$(\lambda_a \circ \lambda_b) \circ \lambda_c = \lambda_{ab} \circ \lambda_c = \lambda_{abc} = \lambda_a \circ (\lambda_b \circ \lambda_c),$$

we show that T ia a semigroup.

For any $\lambda_a \in T$, there exists $\lambda_{a^0} \in T$ satisfied:

$$\lambda_a \circ \lambda_{a^0} \circ \lambda_a = \lambda_{aa^0} \circ \lambda_a = \lambda_{aa^0a} = \lambda_a.$$

Thus we show that T is a regular semigroup.

Then we have the following theorem:

Theorem 2.4 Let S be a regular semigroup with an inverse transversal S^0, and $e = aa^0,\ f = a^0a$ for each $a \in S$. Then
(1) $(e\Lambda e^0)\lambda_a = \Lambda f$, $(f^0If)\lambda_a^{-1} = eI$;
(2) $(eI)\lambda_a = f^0If$, $(\Lambda f)\lambda_a^{-1} = e\Lambda e^0$.

Proof. (1) For any $m, n \in e\Lambda e^0$, there exist $s, t \in \Lambda$, such that $m = ese^0, n = ete^0$.
Then

$$m^2 = ese^0ese^0 = ese^0se^0 = ese^0 = m,\ mn = ese^0ete^0 = ese^0te^0 = este^0 \in e\Lambda e^0.$$

Thus $e\Lambda e^0$ is a band.

Since $e\Lambda e^0 = e\Lambda e^0e \subseteq E(eCe) \subseteq \langle\langle e\rangle\rangle$, thus $e\Lambda e^0$ is a subband of $\langle\langle e\rangle\rangle$.

Similarly, f^0If is a subband of $\langle\langle f\rangle\rangle$.

For any $x \in \Lambda$,

$$(exe^0)\lambda_a = (aa^0xa^{00}a^0)\lambda_a = a^0xa^{00}a^0a = a^0xa^{00}f,$$

and

$$(a^0xa^{00})^0a^0xa^{00} = a^0x^0a^{00}a^0xa^{00} = a^0x^0xa^{00} = a^0xa^{00} \in \Lambda,$$

then we have $(exe^0)\lambda_a \in \Lambda f$, thus $(e\Lambda e^0)\lambda_a \subseteq \Lambda f$.

Conversely, for any $y \in \Lambda$, we know that

$$yf = ya^0a = a^0a^{00}ya^0a = a^0ea^{00}ya^0e^0a,$$

and

$$(a^{00}ya^0)^0a^{00}ya^0 = a^{00}y^0a^0a^{00}ya^0 = a^{00}y^0ya^0 = a^{00}ya^0 \in \Lambda,$$

then we have $yf = (ea^{00}ya^0e^0)\lambda_a \in (e\Lambda e^0)\lambda_a$, thus $\Lambda f \subseteq (e\Lambda e^0)\lambda_a$.

We obtain that $(e\Lambda e^0)\lambda_a = \Lambda f$. Similarly, we can show that $(f^0If)\lambda_a^{-1} = eI$.

(2) For any $g \in I$, we have

$$(eg)\lambda_a = a^0ega = a^0ga = a^0a^{00}a^0ga = a^0a^{00}a^0ga^{00}a^0a = f^0a^0ga^{00}f,$$

and

$$a^0ga^{00}(a^0ga^{00})^0 = a^0ga^{00}a^0g^0a^{00} = a^0gg^0a^{00} = a^0ga^{00} \in I.$$

Thus we obtain that $(eI)\lambda_a \subseteq f^0If$.

Conversely, for any $h \in I$, we have

$$f^0hf = a^0a^{00}ha^0a = a^0ea^{00}ha^0a,$$

and

$$a^{00}ha^0(a^{00}ha^0)^0 = a^{00}ha^0a^{00}h^0a^0 = a^{00}hh^0a^0 = a^{00}ha^0 \in I,$$

so $f^0hf = (ea^{00}ha^0)\lambda_a \in (eI)\lambda_a$, we obtain that $f^0If \subseteq (eI)\lambda_a$.

Thus $(eI)\lambda_a = f^0If$. Similarly, we can show that $(\Lambda f)\lambda_a^{-1} = e\Lambda e^0$.

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Received: January, 2012