Invariance Condition and Amenability
of Locally Compact Groups

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Abstract
In this paper, it is proved that a locally compact group $G$ is amenable if and only if for any measurable subsets $E_1, E_2, \ldots, E_n$ of $G$, $a_1, a_2, \ldots, a_n \in G$ and $\varepsilon > 0$, there exists a compact subset $F$ of $G$ with $\lambda(F) > 0$ such that
\[
| \lambda(a_k^{-1}E_k \cap F) - \lambda(E_k \cap F) | < \varepsilon \lambda(F),
\]
for $k = 1, 2, \ldots, n$, where $\lambda$ denotes the left Haar measure on $G$.

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1 Introduction

In [1], A. Adler and J. Hamilton showed that a discrete semigroup $S$ is left amenable if and only if it satisfies the following invariance condition:

(IC) For any subsets $E_1, E_2, \ldots, E_n$ of $S$ and any $s_1, s_2, \ldots, s_n$ in $S$, there exists a nonempty finite subset $T$ of $S$ such that
\[
n(s_k^{-1}E_k \cap T) = n(E_k \cap T),
\]
for $k = 1, 2, \ldots, n$, where $s^{-1}E = \{ t \in S : st \in E \}$ and $n(E)$ is the number of elements in $E$.

As it is mentioned in [1] for discrete semigroups, the (IC) is equivalent to the following weaker condition.

(WIC) For any subsets $E_1, E_2, \ldots, E_n$ of $S$, any $s_1, s_2, \ldots, s_n \in S$ and any $\varepsilon > 0$, there exists a nonempty finite subset $T$ of $S$ such that
\[
| n(s_k^{-1}E_k \cap T) - n(E_k \cap T) | < \varepsilon n(T),
\]
Thus, perhaps, (WIC) instead of (IC) may be called the invariance condition even in the discrete semigroup situation.

In this paper, first we consider an ultrafilter for an arbitrary nonempty set $X$, and state some facts for group $G$ instead of $X$. Also, we consider the quotient Banach algebra $L_\infty(G)$ with quotient norm $\| \cdot \|_\infty$. Among other results, we generalize results in [1] to topological groups and prove that a locally compact group $G$ is amenable if and only if for any $\lambda$-measurable subsets $E_1, E_2, ..., E_n$ of $G$, any $a_1, a_2, ..., a_n \in G$ and any $\varepsilon > 0$, there exists a compact subset $F$ of $G$ with $\lambda(F) > 0$ such that

$$| \lambda(a_k^{-1}E_k \cap F) - \lambda(E_k \cap F) | < \varepsilon \lambda(F),$$

for $k = 1, 2, ..., n$, where $\lambda$ denotes the left Haar measure on $G$.

2 Preliminaries

In this section, we offer some results which are useful in the sequel.

**Definition 1** Let $G$ be a locally compact group. We say that $G$ has (GIC) property, if it satisfies the following condition:

\[(\text{GIC}) \quad \text{For any } \lambda\text{-measurable subsets } E_1, E_2, ..., E_n \text{ of } G, \text{ any } a_1, a_2, ..., a_n \in G \text{ and any } \varepsilon > 0, \text{ there exists a compact subset } F \text{ of } G \text{ with } \lambda(F) > 0 \text{ such that}

$$| \lambda(a_k^{-1}E_k \cap F) - \lambda(E_k \cap F) | < \varepsilon \lambda(F),$$

for $k = 1, 2, ..., n$, where $\lambda$ denotes the left Haar measure on $G$.

**Remark 1.** Note that group invariance condition (GIC) differs from the classical Følner’s (Følner type) condition. Classical Følner’s condition is based on the measure of difference of sets but here the difference of the measures of sets considered. For more details see [4], [6] and [7].

We recall that an ultrafilter on an arbitrary nonempty set $X$ is a family $\mathcal{X}$ of subsets of $X$ satisfying the following properties:

(a) For $A, B \subseteq X$, if $A \in \mathcal{X}$ and $A \subseteq B$, then $B \in \mathcal{X}$.

(b) For $A, B \in \mathcal{X}$, $A \cap B \in \mathcal{X}$, i.e. $\mathcal{X}$ is closed under finite intersections.

(c) For $A \subseteq X$, exactly one of $A$ and $A^c = X - A$ is in $\mathcal{X}$.

(d) $\emptyset \notin \mathcal{X}$. 
Furthermore, a collection $\mathcal{X}$ of subsets of $X$ can be extended to an ultrafilter if and only if the intersections of finitely many members of $\mathcal{X}$ are nonempty, see [2] and [8] for more details.

Let $\mathbb{R}^X$ be the set of all real valued functions on the nonempty set $X$, and $\mathcal{X}$ be an ultrafilter on $X$. An ultrafilter $\mathcal{X}$ induces an equivalence relation $\sim_{\mathcal{X}}$ on $\mathbb{R}^X$. For $\varphi$ and $\psi$ in $\mathbb{R}^X$, we define the relation $\sim_{\mathcal{X}}$ by

$$\varphi \sim_{\mathcal{X}} \psi \iff \{ x \in X : \varphi(x) = \psi(x) \} \in \mathcal{X}. \quad (2.1)$$

It is clear that $\sim_{\mathcal{X}}$ is an equivalence relation on $\mathbb{R}^X$. Let $\tilde{\mathcal{X}}$ be the family of equivalence classes for $\mathbb{R}^X$ which induced by equivalence relation $\sim_{\mathcal{X}}$ on $\mathbb{R}^X$. Further, the quotient set $\mathbb{R}^X / \mathcal{X}$ forms an ordered field under the following order relation,

$$[\varphi] \prec [\psi] \iff \{ x \in X : \varphi(x) < \psi(x) \} \in \mathcal{X}, \quad (2.2)$$

and the field operations

$$[\varphi] + [\psi] = [\varphi + \psi], \quad (2.3)$$

and

$$[\varphi][\psi] = [\varphi \psi], \quad (2.4)$$

for $\varphi, \psi \in \mathbb{R}^X$, where $\varphi + \psi$ and $\varphi \psi$ are defined pointwise and $[\varphi]$ represents the equivalence class of $\varphi$ under the equivalence relation $\sim_{\mathcal{X}}$ defined by (2.1). We say that $[\varphi] \preceq [\psi]$, when $\varphi \sim_{\mathcal{X}} \psi$ or $[\varphi] \prec [\psi]$.

It is easy to see that $\mathbb{R}$ can be embedded into $\mathbb{R}^X / \mathcal{X}$ as an ordered subfield via $\alpha \mapsto [\alpha.1]$, where $\alpha.1$ is the constant function on $X$ with value $\alpha$. For other results and details for $\mathbb{R}^X / \mathcal{X}$, see [1] and [8].

**Definition 2** An element $[\varphi]$ in $\mathbb{R}^X / \mathcal{X}$ is said to be finite, if there exist real numbers $\alpha, \beta$ in $\mathbb{R}$ such that $[\alpha.1] \preceq [\varphi] \preceq [\beta.1]$ in $\mathbb{R}^X / \mathcal{X}$.

**Remark 2.** We note that $\mathbb{R}$ can be embedded into $\mathbb{R}^X / \mathcal{X}$. So, a real number $\alpha$ can be considered as an element in $\mathbb{R}^X / \mathcal{X}$. Therefore, we put $\alpha \leq \varphi \leq \beta$ instead of $[\alpha.1] \preceq [\varphi] \preceq [\beta.1]$.

**Definition 3** For finite element $[\varphi]$ in $\mathbb{R}^X / \mathcal{X}$, we define the standard part of $[\varphi]$ (s.p.$[\varphi]$) by

$$\text{s.p.}[\varphi] = \sup \{ \alpha \in \mathbb{R} : \alpha < \varphi \}. \quad (2.5)$$

We note that

$$\text{s.p.}[\varphi] = \sup \{ \alpha \in \mathbb{R} : \alpha < \varphi \} = \sup \{ \alpha \in \mathbb{R} : \{ x \in X : \alpha.1(x) < \varphi(x) \} \in \mathcal{X} \}. \quad (2.6)$$
It is easy to see that
\[
\text{s.p.}[\varphi] = \inf \{ \alpha \in \mathbb{R} : \varphi < \alpha \}
\]
\[
= \inf \{ \alpha \in \mathbb{R} : \{ x \in X : \varphi(x) < \alpha \} \in \mathcal{X} \}. 
\]
(2.7)

Lemma 1 Let [\varphi] and [\psi] be finite elements of \( \mathbb{R}^X \). Then,
(a) [\varphi] + [\psi] and [\psi] are finite.
(b) s.p.[\varphi + \psi] = s.p.[\varphi] + s.p.[\psi].
(c) s.p.[\varphi \psi] = (s.p.[\varphi])(s.p.[\psi]).

Proof. The statements (a) and (b) are clear. We prove (c). First, we assume that 0 \( \preceq \) [\varphi], 0 \( \preceq \) [\psi] and s.p.[\varphi] = s.p.[\psi] = 0. Then by (2.4), we have 0 \( \preceq \) [\varphi] and hence
\[
\text{s.p.}[\varphi \psi] \geq 0 = (\text{s.p.}[\varphi])(\text{s.p.}[\psi]). 
\]
(2.8)

On the other hand, for any \( \varepsilon > 0 \), there exist \( \alpha \) and \( \beta \) in \( \mathbb{R} \) such that
\[
0 = \text{s.p.}[\varphi] < \alpha < \text{s.p.}[\varphi] + \varepsilon^\frac{1}{2} = \varepsilon^\frac{1}{2},
\]
and
\[
0 = \text{s.p.}[\psi] < \beta < \text{s.p.}[\psi] + \varepsilon^\frac{1}{2} = \varepsilon^\frac{1}{2},
\]
and \( \varphi < \alpha, \psi < \beta \). Thus, \( \varphi \psi < \alpha \beta \) and so
\[
[\varphi \psi] < [\alpha \beta.1] = [(\alpha.1)(\beta.1)] < \varepsilon.
\]
So by Definition 3, s.p.\([\varphi \psi] \leq 0 \). Hence, our statement is proved in the special case with 0 \( \preceq \) [\varphi], 0 \( \preceq \) [\psi] and s.p.[\varphi] = s.p.[\psi] = 0.

Next, suppose that \([\varphi]\) is finite and \( \alpha > 0 \), it follows by Definition 3 that s.p.\([\alpha \varphi] = \alpha(\text{s.p.}[\varphi]) \). This equality certainly holds for \( \alpha = 0 \) as well.

Finally, if \( \alpha < 0 \), then
\[
0 = \text{s.p.}[0] = \text{s.p.}[0.1] = \text{s.p.}[(\alpha + (-\alpha))\varphi] = \text{s.p.}[\alpha \varphi] + \text{s.p.}[-\alpha \varphi],
\]
which implies that
\[
\text{s.p.}[\alpha \varphi] = -\text{s.p.}[-\alpha \varphi] = -(-\alpha)\text{s.p.}[\varphi] = \alpha(\text{s.p.}[\varphi]).
\]
Now, for the general case, let \([\varphi]\) and \([\psi]\) be finite. Then, we have
\[
\text{s.p.}([\varphi] - \text{s.p.}[\varphi]) = \text{s.p.}([\psi] - \text{s.p.}[\psi]) = 0.
\]
Now, We apply the first situation to each of the following subcases:
(i) \([\varphi] - \text{s.p.}[\varphi] \geq 0 \) and \([\psi] - \text{s.p.}[\psi] \geq 0 \),
(ii) s.p.\([\varphi] - [\varphi] \geq 0 \) and \([\psi] - \text{s.p.}[\psi] \geq 0 \),
Invariance condition

(iii) \([\varphi] - \text{s.p.}[\varphi] \geq 0\) and \(\text{s.p.}[\psi] - [\psi] \geq 0\),
(iv) \(\text{s.p.}[\varphi] - [\varphi] \geq 0\) and \(\text{s.p.}[\psi] - [\psi] \geq 0\).

Our result follows. □

Let \(G\) be a locally compact group and \(\lambda\) be a fixed left Haar measure on \(G\). Let \(BM(G)\) be the algebra of all bounded real valued \(\lambda\)-measurable functions on \(G\) with pointwise operations and supremum norm \(\|\cdot\|_\infty\), and \(\mathcal{N}\) be the set of all locally \(\lambda\)-null functions in \(BM(G)\). Then, \(\mathcal{N}\) is a closed ideal in \(BM(G)\), see [5]. Let \(L_\infty(G) = \frac{BM(G)}{\mathcal{N}}\) be the quotient Banach algebra with quotient norm \(\|\cdot\|_\infty\), i.e., the essential supremum norm. As in [5], we write \(\int f(y)dy\) for \(\int fd\lambda\). A linear functional \(\Gamma\) in \(L_1(G) \hookrightarrow L_\infty(G)^*\) is a mean on \(L_\infty(G)\) if \(\text{ess.inf} f \leq \Gamma(f) \leq \text{ess.sup} f\) for all \(f \in L_\infty(G)\). A mean \(\Gamma\) on \(L_\infty(G)\) is a left invariant mean (LIM), if

\[
\Gamma(\ell_a f) = \Gamma(f),
\]

for any \(a \in G\) and \(f \in L_\infty(G)\), where \(\ell_a f \in L_\infty(G)\) is well defined via

\[
(\ell_a f)(b) = f(ab), \quad b \in G.
\]

A mean \(\Gamma\) on \(L_\infty(G)\) is called a topological left invariant mean (TLIM), if

\[
\Gamma(\varphi \ast f) = \Gamma(f),
\]

for any \(f \in L_\infty(G)\) and \(\varphi \in \mathcal{P}(G) = \{\psi \in L_1(G) : \psi \geq 0 \text{ and } \|\psi\|_1 = 1\}\), where the convolution \(\ast\) is defined by

\[
\varphi \ast f(x) = \int_G \varphi(y) f(y^{-1}x)dy,
\]

for more details see [5].

It is well known that every topological left invariant mean is a left invariant mean. Also, \(L_\infty(G)\) has a topological left invariant mean if and only if it has a left invariant mean. When \(L_\infty(G)\) has a left invariant mean, and hence a topological left invariant mean, we say that \(G\) is (left) amenable, see [4], [6] and [7].

3 Main Results

Let \(G\) be a locally compact group and \(\lambda\) be a fixed left Haar measure on \(G\). We put

\[
X = \{F \subseteq G : F \text{ is compact and } \lambda(F) > 0\}.
\]
For $f \in \mathcal{L}_\infty(G)$, we define
\[ \overline{f}(F) = \frac{1}{\lambda(F)} \int_F f \, d\lambda, \quad F \in X. \] (3.2)
Then $\overline{f} : X \to \mathbb{R}$ is well defined, since $f = g \lambda$ a.e. implies that for any compact set $F \subseteq G$,
\[ \int_F f \, d\lambda = \int_F g \, d\lambda. \]
If $\mathcal{X}$ is an ultrafilter on $X$, then it is easily checked that the map $\gamma : \mathcal{L}_\infty(G) \to \mathbb{R}^X$ defined by
\[ \gamma(f) = [f], \] (3.3)
for $f \in \mathcal{L}_\infty(G)$ is linear and belongs to $\mathbb{R}^X$. We have
\[ \text{ess.inf } f \leq \gamma(f) \leq \text{ess.sup } f, \] (3.4)
for all $f \in \mathcal{L}_\infty(G)$. Therefore, the standard part $\gamma(f)$ is well defined. Put
\[ \Gamma(f) = s.p.(\gamma(f)) = s.p.\overline{f}. \] (3.5)
It is easy to see that $\Gamma : \mathcal{L}_\infty(G) \to \mathbb{R}$ is a mean on $\mathcal{L}_\infty(G)$. Hence, we have:

**Lemma 2** Let $G$ be a locally compact group and $\lambda$ be a fixed left Haar measure on $G$. Let $X$ be defined by (3.1). Then any ultrafilter $\mathcal{X}$ on $X$ gives rise to a mean on $\mathcal{L}_\infty(G)$.

We know that in the discrete semigroup situation, means that vanish on singletons come from ultrafilters via the above construction [1, p.75]. For a non-discrete locally compact group of $G$, however, all of the means on $\mathcal{L}_\infty(G)$ come from ultrafilters.

Let $G$ be a non-discrete locally compact group, i.e., the locally compact group with a non-discrete Hausdorff topology, and $\Gamma$ be a mean on $\mathcal{L}_\infty(G)$. For any measurable set $E \subseteq G$, let $\xi_E \in \mathcal{L}_\infty(G)$ denote the characteristic function of $E$. For convenience, we write $\Gamma(E) = \Gamma(\xi_E)$, where $\Gamma : \mathcal{L}_\infty(G) \to \mathbb{R}$ is a mean.

**Theorem 1** Let $G$ be a non-discrete locally compact group and $\lambda$ be a fixed left Haar measure on $G$ and $X = \{ F \subseteq G : F \text{ is compact and } \lambda(F) > 0 \}$. For each $F \in X$, we put
\[ \mathcal{N}_F(E) = \overline{\xi_E}(F) = \frac{\lambda(E \cap F)}{\lambda(F)}, \] (3.6)
for any $\lambda$-measurable set $E \subseteq G$. Also, for given $\varepsilon > 0$, we define
\[ \mathcal{B}_\varepsilon(E) = \{ F \in X : | \mathcal{N}_F(E) - \Gamma(F) | < \varepsilon \}. \] (3.7)
Then the finite intersections of sets of the form (3.7) are nonempty.
Proof. For \( \lambda \)-measurable subsets \( E_1, E_2, \ldots, E_n \) of \( G \) and \( \varepsilon > 0 \), we can construct \( k \) mutually disjoint measurable sets \( U_1, U_2, \ldots, U_k \) such that they form a partition of \( G \) and that each \( E_i \) is the union of no more than \( 2^{n-1} \) sets of \( U_j \)'s where \( k \leq 2^n \), [3, Lemma 1, p.71]. Without loss of generality, we may assume that

\[
\Gamma(U_j) = \Gamma(\xi_{U_j}) > 0, \quad 1 \leq j \leq m, \tag{3.8}
\]

\[
\Gamma(U_j) = \Gamma(\xi_{U_j}) = 0, \quad m + 1 \leq j \leq k, \tag{3.9}
\]

for some integer \( m \). Since \( U_j \)'s form a partition of \( G \), so

\[
\sum_{j=1}^{m} \Gamma(U_j) = \sum_{j=1}^{m} \Gamma(\xi_{U_j}) = \sum_{j=1}^{k} \Gamma(\xi_{U_j}) = \sum_{j=1}^{k} \Gamma(U_j) = \Gamma(G) = 1.
\]

Using the density of the rational numbers in \( \mathbb{R} \), we can find rational numbers \( p_1, p_2, \ldots, p_k \) such that

(i) \( p_j > 0 \) for \( 1 \leq j \leq m \) and \( p_j = 0 \) for \( m + 1 \leq j \leq k \);

(ii) \( | p_j - \Gamma(U_j) | < \frac{\varepsilon}{2^{n-1}} \) for \( 1 \leq j \leq k \);

(iii) \( \sum_{j=1}^{m} p_j = \sum_{j=1}^{k} p_j = 1 \).

For \( i = 1, 2, \ldots, m \),

\[
\text{ess.sup } \xi_{U_j} \geq \Gamma(\xi_{U_j}) = \Gamma(U_j) > 0.
\]

Thus, \( U_j \) is not locally \( \lambda \)-null. Hence, there exists a compact set \( K_j \subseteq U_j \) such that \( \lambda(K_j) > 0 \). Choose \( \alpha > 0 \) so large that

\[
0 < \frac{p_j}{\alpha} < \lambda(K_j), \quad j = 1, 2, \ldots, m.
\]

Since \( G \) is non-discrete locally compact group then \( \lambda(A) = 0 \) for all countable subset \( A \) of \( G \), i.e., the left Haar measure \( \lambda \) is continuous measure, [5, 15.17(b), p.198], and hence by [5, 11.44(a), p.132] there exists a compact set \( F_j \), which \( F_j \subseteq K_j \subseteq U_j \) and

\[
\lambda(F_j) = \frac{p_j}{\alpha} > 0, \quad j = 1, 2, \ldots, m.
\]

For \( j = m + 1, \ldots, k \), put \( F_j = \emptyset \subseteq U_j \). Therefore,

\[
\lambda(F_j) = 0 = \frac{p_j}{\alpha}, \quad j = m + 1, \ldots, k.
\]

Now, we put \( F = \bigcup_{j=1}^{k} F_j = \bigcup_{j=1}^{m} F_j \). Thus \( F \) is a compact subset of \( G \) and

\[
\lambda(F) = \sum_{j=1}^{m} \lambda(F_j) = \sum_{j=1}^{m} \frac{p_j}{\alpha} = \frac{1}{\alpha} > 0,
\]
since the $U_j$’s are mutually disjoint and each $F_j \subseteq U_j$. Hence, $F$ belongs to $X$.

For $i = 1, 2, ..., n$, we have $E_i = U_{i_1} \cup ... \cup U_{i_r}$ where $r \leq 2^{n-1}$. Since the $U_j$’s are mutually disjoint, from the definition $F$, it follows that

$$F \cap U_j = F_j, \quad j = 1, 2, ..., k.$$ 

Hence,

$$N_F(E_i) = \frac{\lambda(E_i \cap F)}{\lambda(F)}$$

$$= \sum_{j=1}^{r} \frac{\lambda(U_{i_j} \cap F)}{\lambda(F)}$$

$$= \sum_{j=1}^{r} \frac{\lambda(F_{ij})}{\lambda(F)}$$

$$= \sum_{j=1}^{r} p_{ij}$$

$$= \sum_{j=1}^{r} \alpha \lambda(F).$$

So,

$$|N_F(E_i) - \Gamma(E_i)| = |\sum_{j=1}^{r} p_{ij} - \sum_{j=1}^{r} \Gamma(U_{i_j})|$$

$$\leq \sum_{j=1}^{r} |r_{ij} - \Gamma(U_{i_j})|$$

$$< \sum_{j=1}^{r} \frac{\epsilon}{2^{n-1}}$$

$$= \frac{r \epsilon}{2^{n-1}} \leq \epsilon.$$ 

Hence, $F \in B_\epsilon(E_i)$ for $i = 1, 2, ..., n$. Thus,

$$\bigcap_{i=1}^{n} B_\epsilon(E_i) \neq \emptyset. \quad (3.10)$$

In the general case, if we have sets $B_{\epsilon_1}(E_1), B_{\epsilon_2}(E_2), ..., B_{\epsilon_n}(E_n)$, then

$$\emptyset \neq \bigcap_{i=1}^{n} B_\epsilon(E_i) \subseteq \bigcap_{i=1}^{n} B_{\epsilon_i}(E_i), \quad i = 1, 2, ..., n,$$

where $\epsilon = \min(\epsilon_1, \epsilon_2, ..., \epsilon_n) > 0$. This shows that finite intersections of sets of the form (3.7) are nonempty, and the proof is complete. □

We know that a collection $\mathcal{X}$ of subsets of $X$ can be extended to an ultrafilter if and only if the intersections of finitely many members of $\mathcal{X}$ are nonempty, [2] and [8]. Hence, with consider to Theorem 1, we prove that there is an ultrafilter $\mathcal{X}$ on $X$ which contains all the sets of the form $B_\epsilon(E)$, where $X$ is defined by (3.1).
Theorem 2 Let $G$ be a non-discrete locally compact group, and $\Gamma \in \mathcal{L}_\infty(G)^*$ be a mean on $\mathcal{L}_\infty(G)$. Then there exists an ultrafilter $\mathcal{X}$ on the set $X = \{F \subseteq G : F$ is compact and $\lambda(F) > 0\}$ such that

$$\Gamma(f) = \text{s.p.}[f],$$

for all $f \in \mathcal{L}_\infty(G)$.

Proof. By Theorem 1, the collection $\mathcal{X}$ contains all of the sets of the form $\mathcal{B}_\varepsilon(E)$ is an ultrafilter on the set $X$. To complete the proof, we show that for all $f \in \mathcal{L}_\infty(G),

$$\Gamma(f) = \text{s.p.}[f].$$

(3.11)

Let $E$ be a measurable subset of $G$ with respect to the left Haar measure $\lambda$ on $G$, and $\varepsilon > 0$. Then,

$$\{F \in X : \Gamma(E) - \varepsilon < \xi_E(F)\} \cap \{F \in X : \xi_E(F) < \Gamma(E) + \varepsilon\}
= \{F \in X : |\xi_E(F) - \Gamma(E)| < \varepsilon\}
= \{F \in X : |N_F(E) - \Gamma(E)| < \varepsilon\}
= B_\varepsilon(E) \in X.$$

By property (a) of an ultrafilter, we have

$$\{F \in X : \Gamma(E) - \varepsilon < \xi_E(F)\} \in \mathcal{X},$$

and

$$\{F \in X : \xi_E(F) < \Gamma(E) + \varepsilon\} \in \mathcal{X}.$$

By definition of standard part of an element in ordered field $\mathbb{R}_X$, we have

$$\Gamma(E) - \varepsilon < \xi_E,$$

(3.12)

and

$$\xi_E < \Gamma(E) + \varepsilon,$$

(3.13)

for any real number $\varepsilon > 0$. From Definition 3, we get

$$\text{s.p.} [\xi_E] = \Gamma(E) = \Gamma(\xi_E),$$

(3.14)

where $E$ is $\lambda$-measurable subset of $G$. By Lemma 2, we know that the standard part of $[f]$ defines a mean on $\mathcal{L}_\infty(G)$, so $\Gamma'(f) = \text{s.p.}[f]$ defines a mean on $\mathcal{L}_\infty(G)$. The identity (3.14) shows that $\Gamma$ and $\Gamma'$ agree on the measurable simple functions. On the other hand, the measurable simple functions are
norm dense in $BM(G)$ and hence in $L_\infty(G)$ as well. Thus, the means $\Gamma$ and $\Gamma'$ agree on $L_\infty(G)$ by continuity. That is

$$\Gamma(f) = \Gamma'(f) = \text{s.p.}[\overline{f}], \quad (3.15)$$

for all $f \in L_\infty(G)$. □

Now, we shall prove that a locally compact group of $G$ is amenable if and only if $G$ satisfies the group invariance condition (GIC).

**Theorem 3** Let $G$ be a non-discrete locally compact group. Then $G$ is amenable if and only if $G$ satisfies (GIC).

**Proof.** Let $G$ be an amenable non-discrete locally compact group. Thus, $L_\infty(G)$ has a left invariant mean $\Gamma$. By Theorem 2, there exists an ultrafilter $\mathcal{X}$ on $X$ such that for all $f \in L_\infty(G)$,

$$\Gamma(f) = \text{s.p.}[\overline{f}],$$

where the set $X$ is defined by (3.1) and

$$\overline{f}(F) = \frac{1}{\lambda(F)} \int_{F} f d\lambda, \quad F \in X.$$ 

Hence, if $E$ is a $\lambda$-measurable subset of $G$ and $a \in G$ then

$$\text{s.p.}[\xi_{a^{-1}E}] = \text{s.p.}[\xi_{aE}]$$

$$= \Gamma(\ell_{aE})$$

$$= \Gamma(\xi_{E})$$

$$= \Gamma(E)$$

$$= \text{s.p.}[\xi_{E}].$$

It follows that in the quotient space $\mathbb{R}_X^X/\mathcal{X}$ for any real number $\varepsilon > 0$,

$$-\varepsilon < \overline{\xi_{E}} - \overline{\xi_{a^{-1}E}},$$

and

$$\overline{\xi_{E}} - \overline{\xi_{a^{-1}E}} < \varepsilon,$$

and hence by property (b) of an ultrafilter, we have

$$\{F \in X : -\varepsilon < \overline{\xi_{E}}(F) - \overline{\xi_{a^{-1}E}}(F)\} \cap \{F \in X : \overline{\xi_{E}}(F) - \overline{\xi_{a^{-1}E}}(F) < \varepsilon\} \in \mathcal{X}. \quad (3.16)$$

Therefore, finite intersections of sets of the form

$$\{F \in X : |\lambda(a^{-1}E \cap F) - \lambda(E \cap F)| < \varepsilon \lambda(E)\},$$
for measurable subsets of $E$ of $G$ are in $\mathcal{X}$ again, since
\[
\{ F \in X : | \lambda(a^{-1}E \cap F) - \lambda(E \cap F) | < \varepsilon \lambda(F) \}
\]
\[
= \{ F \in X : \left| \frac{\lambda(a^{-1}E \cap F)}{\lambda(F)} - \frac{\lambda(E \cap F)}{\lambda(F)} \right| < \varepsilon \}
\]
\[
= \{ F \in X : | f_x \xi_{a^{-1}E} d\lambda - f_x \xi_{E} d\lambda \lambda(F) | < \varepsilon \}
\]
\[
= \{ F \in X : | \xi_{a^{-1}E}(F) - \xi_{E}(F) | < \varepsilon \}
\]
\[
= \{ F \in X : -\varepsilon < \xi_{E}(F) - \xi_{a^{-1}E}(F) < \varepsilon \}
\]
\[
\cap \{ F \in X : | \xi_{a^{-1}E}(F) - \xi_{a^{-1}E}(F) | < \varepsilon \}
\]
which is in $\mathcal{X}$, by (3.16). Now, by property (d) of an ultrafilter, these intersections are nonempty. This implies (GIC).

Conversely, suppose $G$ satisfies (GIC). For any $\lambda$-measurable subset $E$ of $G$, $a \in G$ and $\varepsilon > 0$, we write
\[
\mathcal{B}_{\varepsilon}(E, a) = \{ F \in X : | \lambda(a^{-1}E \cap F) - \lambda(E \cap F) | < \varepsilon \lambda(F) \},
\]
where $X = \{ F \subseteq G : F$ is compact and $\lambda(F) > 0 \}$. Group invariance condition (GIC) implies that finite intersections of sets of the form $\mathcal{B}_{\varepsilon}(E, a)$ are nonempty. Hence, there exists an ultrafilter $\mathcal{X}$ on $X$ containing all the $\mathcal{B}_{\varepsilon}(E, a)$’s. Thus,
\[
\{ F \in X : -\varepsilon < \xi_{E}(F) - \xi_{a^{-1}E}(F) \} \cap \{ F \in X : | \xi_{E}(F) - \xi_{a^{-1}E}(F) | < \varepsilon \}
\]
\[
= \{ F \in X : | \xi_{E}(F) - \xi_{a^{-1}E}(F) | < \varepsilon \}
\]
\[
= \{ F \in X : | f_x \xi_{E} d\lambda - f_x \xi_{a^{-1}E} d\lambda \lambda(F) | < \varepsilon \}
\]
\[
= \{ F \in X : | \lambda(E \cap F) - \lambda(a^{-1}E \cap F) | < \varepsilon \}
\]
\[
= \{ F \in X : \lambda(E \cap F) - \lambda(a^{-1}E \cap F) | < \varepsilon \}
\]
\[
= \mathcal{B}_{\varepsilon}(E, a) \in \mathcal{X}.
\]
By property (a) for an ultrafilter and the definition of $\prec$ in the ordered field $\mathbb{R}^\mathcal{X}$ by (2), we have
\[
-\varepsilon < \xi_{E} - \xi_{a^{-1}E},
\]
and
\[
\xi_{E} - \xi_{a^{-1}E} < \varepsilon,
\]
Taking standard parts, we have
\[
s.p.\xi_{E} = s.p.\xi_{a^{-1}E} = s.p.\xi_{E},
\]
for any \( \lambda \)-measurable subset \( E \) of \( G \) and any \( a \in G \). Using norm density of the \( \lambda \)-measurable simple functions in \( L_\infty(G) \), the mean \( \Gamma \) on \( L_\infty(G) \) defined by \( \Gamma(f) = \text{s.p.}[f] \) is left invariant and so \( G \) is amenable. \( \square \)

**Remark 3.** If \( G \) is discrete locally compact group, then the left Haar measure \( \lambda \) on \( G \) is \( \lambda(E) = n(E) \), for more details see [5, 15.17 (a) and 15.17 (b)]. Also, every subsets of \( G \) are \( \lambda \)-measurable and \( L_\infty(G) = m(G) \). Thus, (GIC) reduces to (WIC). Therefore, our statement in Theorem 3.4 is just result in [1] applied to groups.

Theorems 1 and 2, relates means to ultrafilter and condition (GIC) is used to exhibit some special ultrafilter so that the mean constructed is left invariant. With this idea, we can set up a condition giving us still other ultrafilters such that the resulting mean is topological left invariant.

**Definition 4** Let \( G \) be a locally compact group and \( \lambda \) be a fixed left Haar measure on \( G \). We say that \( G \) has property of topological invariance condition (TIC), if it satisfies the following condition:

\[(\text{TIC}): \text{For any } \lambda\text{-measurable subsets } E_1, E_2, \ldots, E_n \text{ of } G, \text{ any } \varphi_1, \varphi_2, \ldots, \varphi_n \text{ in } \mathcal{P}(G) = \{ \psi \in L_1(G) : \psi \geq 0 \text{ and } \| \psi \|_1 = 1 \} \text{ and any } \varepsilon > 0, \text{ there exists } F \in X \text{ such that}\]

\[| \lambda(E_k \cap F) - \int_G \varphi_k(y) \lambda(yE_k \cap F) \, dy | < \varepsilon \lambda(F),\]

for \( k = 1, 2, \ldots, n \), where \( X \) is given by (3.1).

**Remark 4.** Similar to Theorem 3, we can conclude that \( L_\infty(G) \) has a topological left invariant mean if and only if it satisfies the topological invariance condition.

**Theorem 4** Let \( G \) be a locally compact group and \( \lambda \) is the left Haar measure on \( G \). The following statements are equivalent.

(a) \( G \) satisfies (GIC).

(b) \( G \) satisfies (TIC).

**Proof.** Let \( G \) is discrete, then \( L_1(G) = \ell_1(G) \) and (TIC) is formally stronger than (GIC). On the other hand, using the facts that \( \lambda(E) = n(E) \) and the \( \ell_1 \)-norm density \( \mathcal{P}_c(G) \) in \( \mathcal{P}(G) \), where

\[\mathcal{P}_c(G) = \{ \psi \in \mathcal{P}(G) : \psi \text{ has finite support} \},\]

we can easily see that (GIC) implies (TIC).

Now, let \( G \) be a non-discrete locally compact group. For any \( \lambda \)-measurable subset \( E \) of \( G \), any \( \psi \in \mathcal{P}(G) \) and any \( \varepsilon > 0 \), we defined

\[\mathcal{B}_\varepsilon(E, \psi) = \{ F \in X : | \lambda(E \cap F) - \int_G \psi(y) \lambda(yE \cap F) \, dy | < \varepsilon \lambda(F) \}.\]
If \( F \in X \), then by (2.12) and (3.6)
\[
\psi \ast \xi_{E}(F) = \frac{1}{\lambda(F)} \int_{F} \psi \ast \xi_{E} d\lambda
\]
\[
= \frac{1}{\lambda(F)} \int_{G} \xi_{E}(x) \int_{G} \psi(y) \xi_{F}(y^{-1}x) dy dx
\]
\[
= \frac{1}{\lambda(F)} \int_{G} \psi(y) \int_{G} \xi_{F}(x) \xi_{yE}(x) dx dy
\]
\[
= \frac{1}{\lambda(F)} \int_{G} \psi(y) \lambda(yE \cap F) dy.
\]
Hence,
\[
B_{\varepsilon}(E, \psi) = \{ F \in X : \lambda(E \cap F) - \int_{G} \psi(y) \lambda(yE \cap F) dy < \varepsilon \lambda(F) \}
\]
\[
= \{ F \in X : | \xi_{E}(F) - \psi \ast \xi_{E}(F) | < \varepsilon \}.
\]
Thus, the proof now works as in Theorem 3, and implies that (GIC) is equivalent to (TIC). \( \square \)

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**References**


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