On Mixed Trilateral Generating Functions
of Modified Jacobi Polynomials

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Abstract

In this article we have obtained some results on the extension of mixed trilateral generating relation involving modified Jacobi polynomials by introducing a linear partial differential operator which does not seem to appear before. Some particular cases of interest are also pointed out.

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1. Introduction

The role of generating functions in the study of special functions is of greater importance. Most of the generating functions available in the literature are almost bilateral in nature. There is dearth of pure trilateral generating relations in the field of special functions. Among the various methods of obtaining generating functions, group-theoretic method seems to be much more potent one in comparison to other methods.
Group-theoretic method in the study of problems of special functions has been started in the mid-fifties of the nineteenth century by L. Weisner [4] while studying generating functions of Hypergeometric Functions. From seventies and onwards (i.e. just after the publication of the book “Obtaining generating functions” by E.B. McBride [2]) of the last century, Weisner’s method has been extensively utilized by the researchers in the derivation of generating functions.

In this paper, we have adopted the group-theoretic method to obtain a novel result on mixed trilateral generating functions of modified Jacobi polynomials, \( P_{n+k}^{(\alpha, \beta)}(x) \) where \( P_{n}^{(\alpha, \beta)}(x) \) is defined by [3]

\[
P_{n}^{(\alpha, \beta)}(x) = \left(\frac{1}{n!}\right) n F_1 \left[ \begin{array}{c} -n, 1+\alpha + \beta + n ; \\ 1+\alpha ; \\ \frac{1-x}{2} \end{array} \right].
\]

(1.1)

In fact, we obtain the following theorem as the main result of our investigation which does not seem to have appeared in the earlier works.

**Theorem:** If there exists a bilateral generating relation of the form:

\[
G(x,u,t) = \sum_{n=0}^{\infty} a_n P_{n+k}^{(\alpha+\sigma, \beta)}(x) g_n(u) t^n,
\]

where \( g_n(u) \) is an arbitrary polynomial of degree \( n \), then

\[
(1+z)^n \left(1+\frac{z}{2}(1-x)\right)^{-1-a-\beta-k} G \left( \begin{array}{c} x-\frac{z}{2}(1-x) \\ 1+\frac{z}{2}(1-x) \end{array} , u , \frac{tz(1+z)}{\left(1+\frac{z}{2}(1-x)\right)^{\frac{1}{2}}} \right)
\]

(1.3)

\[
= \sum_{n=0}^{\infty} z^n \sigma_n(x,u,t),
\]

where
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\[ \sigma_n(x,u,t) = \sum_{p=0}^{n} a_p \binom{n+k}{p+k} P_{n+k}^{(\alpha-n+\beta)}(x) g_p(u) t^p. \]

The importance of the above theorem lies in the fact that whenever one knows a bilateral generating relation of the type (1.2), the corresponding mixed trilateral generating relation can at once be written down from the relation (1.3). Thus one can get a large number of mixed trilateral generating relation from (1.3) by attributing different values to \( a_n \) in (1.2).

2. Derivation of the operator and its extended form of the group

Let us consider the following first order linear partial differential operator,

\[ R = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_0. \]

such that

\[ R(P_{n+k}^{(\alpha+n, \beta)}(x)y^\alpha z^n) = a_n P_{n+k+1}^{(\alpha+n+1, \beta)}(x) y^{\alpha+2} z^{n+1} \]

where \( A_i \) (\( i = 0, 1, 2, 3 \)) are functions of \( x, y, z \) but independent of \( n, \alpha \) and \( a_n \) is a function of \( n, \alpha, \beta \) but independent of \( x, y, z \).

Now using the following differential recurrence relation [3]:

\[ \frac{d}{dx}(P_{n}^{(\alpha, \beta)}(x)) = \frac{1}{1-x^2} \left[ (n+\alpha+\beta+1)(x-1)+2\alpha \right] P_{n}^{(\alpha, \beta)}(x) - 2(n+1) P_{n+1}^{(\alpha, \beta)}(x) \]

we obtain

\[ R = (1-x^2)y^{-2} \frac{\partial}{\partial x} - (x+1)y^{-1} \frac{\partial}{\partial y} - 2xy^{-2} z^2 \frac{\partial}{\partial z} - (1+\beta+k)(x-1)y^{-2} z \]

such that

\[ R(P_{n+k}^{(\alpha+n, \beta)}(x) y^\alpha z^n) = -2(n+k+1) P_{n+k+1}^{(\alpha+n+1, \beta)}(x) y^{\alpha+2} z^{n+1}. \]
Let $\phi(x, y, z)$ be a function such that $R\phi(x, y, z) = 0$. Then on solving $R\phi = 0$, we get a solution as $\phi = (1 + x)^{-\beta-1} yz^{-1}$.

Let us transform the operator $R$ to $E$:

$$E = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}$$

then

$$E = \phi^{-1}(x, y, z) R \phi(x, y, z)$$

i.e.

$$R = \phi(x, y, z) E \phi^{-1}(x, y, z)$$

Let $X, Y, Z$ be a set of new variables for which

$$(2.6) \quad EX = 1, \quad EY = 0, \quad EZ = 0$$

so that $E$ reduce to $D = \frac{\partial}{\partial X}$.

Now solving (2.6), we get

$$X = \frac{y^2}{(x-1)z}, \quad Y = \frac{1-x}{y}, \quad Z = \frac{1-x^2}{z}$$

from which we get

$$x = -\left(\frac{Z}{XY^2} + 1\right), \quad \frac{y}{X} = \left(\frac{2XY^2 + Z}{XY^3}\right), \quad z = -\left(\frac{Z}{X^2Y^4} + \frac{2}{X^2Y^2}\right)$$

Then

$$e^{wR}f(x, y, z) = \phi(x, y, z) e^{wD}f(X, Y, Z)$$

$$= \phi(x, y, z) F(X + w, Y, Z)$$

$$= \phi(x, y, z) g(x, y, z)$$

Supposing that $F(X + w, Y, Z)$ is transformed into $g(x, y, z)$ by reverse substitution.

On calculation, we get

$$(2.6) \quad e^{wR}f(x, y, z) = \left[1 - (1-x)y^{-2}zw\right]^{-\beta-1}$$

$$\times f\left(\frac{x + (1-x)y^{-2}zw}{1 - (1-x)y^{-2}zw}, \frac{y(1-2y^{-2}zw)}{1 - (1-x)y^{-2}zw}, \frac{z(1-2y^{-2}zw)}{1 - (1-x)y^{-2}zw}\right)$$

Now we notice that
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\begin{equation}
(2.7) \quad e^{wR} \left( P_{n+k}^{(\alpha+n, \beta)}(x) y^n z^n \right) = y^n z^n \left( 1 - 2y^{-2}zw \right)^{\alpha+n} \left( 1 - (1-x)y^{-2}zw \right)^{-1-\alpha-\beta-2n-k} \times P_{n+k}^{(\alpha+n, \beta)} \left( \frac{x + (1-x)y^{-2}zw}{1 - (1-x)y^{-2}zw} \right).
\end{equation}

Again,

\begin{equation}
(2.8) \quad e^{wR} \left( P_{n+k}^{(\alpha+n, \beta)}(x) y^n z^n \right) = y^n z^n \sum_{p=0}^{\infty} \frac{(n + k + 1)_p}{p!} P_{n+k+p}^{(\alpha+n-p, \beta)}(x) (-2y^{-2}zw)^p.
\end{equation}

Equating (2.7) and (2.8) and then replacing \(-2y^{-2}zw\) by \(t\), we get

\begin{equation}
(2.9) \quad (1+t)^{\alpha+n} \left[ 1 + \frac{t}{2} (1-x) \right]^{-1-\alpha-\beta-k-2n} \sum_{p=0}^{\infty} \frac{(n + k + 1)_p}{p!} P_{n+k+p}^{(\alpha+n-p, \beta)}(x) t^p.
\end{equation}

3. Proof of the theorem

Now the right hand side of (1.3)

\begin{align*}
&= \sum_{n=0}^{\infty} z^n \sigma_n(x,u,t) \\
&= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} \binom{n+k}{p+k} \sum \left[ \begin{array}{l} (n + k)_{p+k} P_{n+k}^{(\alpha+n+2p, \beta)}(x) g_p(u) t^p \\
\text{[using (1.4)]}
\end{array} \right] \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} z^{n+p} \binom{n+k}{p+k} \sum P_{n+k+p}^{(\alpha+n+p, \beta)}(x) g_p(u) t^p \\
&= \sum_{p=0}^{\infty} a_p (tiz)^p \sum_{n=0}^{\infty} \binom{n+k}{p+k} P_{n+k+p}^{(\alpha+n+p, \beta)}(x) z^n g_p(u)
\end{align*}
This completes the proof of the theorem.

Now we would like to point it out that theorem-1 can be proved as follows by the direct application of the operator R by using the method as discussed in [1].

Let

$\sum_{n=0}^{\infty} a_n \left[ P_{n+k}^{(\alpha, \beta)}(x) y^{\alpha} z^{n} \right] g_n(u) t^n$.

Replacing $t$ by $tz$ in (3.2) and then multiplying both sides of the same by $y^{\alpha}$, we get

$\sum_{n=0}^{\infty} a_n \left[ P_{n+k}^{(\alpha, \beta)}(x) y^{\alpha} z^{n} \right] g_n(u) t^n$.

Now operating $(\exp wR)$ on both sides of (3.3), we get

$(\exp wR)(y^{\alpha} G(x, u, tz)) = (\exp wR) \left( \sum_{n=0}^{\infty} a_n \left[ P_{n+k}^{(\alpha, \beta)}(x) y^{\alpha} z^{n} \right] g_n(u) t^n \right)$. 
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The left member of (3.4), with the help of (2.6), becomes

\[
(3.5) \quad y^\alpha (1 - 2 y^{-2} zw)^\alpha \left[ 1 - (1 - x) y^{-2} zw \right]^{1-\alpha-\beta-k} \times G \left( \frac{x + (1 - x) y^{-2} zw}{1 - (1 - x) y^{-2} zw}, \quad u, \quad \frac{tz (1 - 2 y^{-2} zw)}{(1 - (1 - x) y^{-2} zw)^2} \right).
\]

The right member of (3.4), with the help of (2.5), becomes

\[
\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n t^n \frac{W^p}{p!} \left( P_{n+k}^{(\alpha+n, \beta)} \right) (x) y^\alpha z^p g_n (u)
= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n t^n \frac{W^p}{p!} (-2)^p (n + k + 1)_p \left( P_{n+k+p}^{(\alpha+n-2p, \beta)} \right) (x) y^{\alpha-2p} z^{n+p} g_n (u)
= y^\alpha \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} z^{n+p} a_n \frac{(n + k + 1)_p}{p!} P_{n+k+p}^{(\alpha+n-2p, \beta)} (x) \left( \frac{-2w}{y^2} \right)^p t^n g_n (u)
= y^\alpha \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} a_{n-p} \frac{(n - p + k + 1)_p}{p!} P_{n+k}^{(\alpha+n-2p, \beta)} (x) \left( \frac{-2w}{y^2} \right)^p t^{n-p} g_{n-p} (u).
\]

Equating the above two results, we get

\[
(1 - 2 y^{-2} zw)^\alpha \left[ 1 - (1 - x) y^{-2} zw \right]^{1-\alpha-\beta-k} G \left( \frac{x + (1 - x) y^{-2} zw}{1 - (1 - x) y^{-2} zw}, \quad u, \quad \frac{tz (1 - 2 y^{-2} zw)}{(1 - (1 - x) y^{-2} zw)^2} \right)
= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} a_p \frac{(p + k + 1)_p}{(n - p)!} P_{n+k}^{(\alpha+n-2p, \beta)} (x) \left( \frac{-2w}{y^2} \right)^p t^{n-p} g_p (u).
\]

Putting \( -2w \left/ y^2 \right. = 1 \), we get

\[
(1 + z)^\alpha \left[ 1 + \frac{z}{2} (1 - x) \right]^{1-\alpha-\beta-k} G \left( \frac{x - \frac{z}{2} (1 - x)}{1 + \frac{z}{2} (1 - x)}, \quad u, \quad \frac{tz (1 + z)}{1 + \frac{z}{2} (1 - x)} \right) = \sum_{n=0}^{\infty} z^n \sigma_n (x, u, t),
\]

where

\[
\sigma_n (x, u, t) = \sum_{p=0}^{n} a_p \frac{(n + k)_p}{p!} \left( P_{n+k}^{(\alpha+n-2p, \beta)} \right) (x) t^p g_p (u).
\]

This completes the proof of the theorem-1.
Corollary: If we put $k = 0$ in the above theorem, we get the following result:

If there exists a bilateral generating relation of the form:

$$G(x, u, t) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha+n, \beta)}(x) g_n(u) t^n,$$

where $g_n(u)$ is an arbitrary polynomial of degree $n$, then

$$\sum_{n=0}^{\infty} z^n \sigma_n(x, u, t),$$

where

$$\sigma_n(x, u, t) = \sum_{p=0}^{n} a_p \binom{n}{p} P_n^{(\alpha+n+2p, \beta)}(x) g_p(u) t^p,$$

which is worthy of notice.

References


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