Some Results on Geometric Mean Graphs

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Abstract

A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to be a Geometric mean graph if it is possible to label the vertices \( x \in V \) with distinct labels \( f(x) \) from 1, 2 \ldots q+1 in such way that when each edge \( e = uv \) is labeled with \( f(uv) = \lfloor \sqrt[2]{f(u)f(v)} \rfloor \) or \( \lceil \sqrt[2]{f(u)f(v)} \rceil \) then the edge labels are distinct. Here we prove that \( C_m \cup P_n, C_m \cup C_n, nK_3, nK_3 \cup P_n, nK_3 \cup C_m \), crown, square of a path are geometric mean graphs.

Keywords: Graph, geometric mean graph, crown, square of a path, union of graphs.

Introduction

The graphs considered here will be finite, undirected and simple graph without isolated vertices. The vertex set is denoted by \( V(G) \) and edge set is denoted by \( E(G) \). The union of two graphs \( G_1 \) and \( G_2 \) has vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \). The join \( G_1 + G_2 \) of two graphs \( G_1 \) and \( G_2 \) has vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1) \) and \( v \in V(G_2)\} \). \( mG \) denotes the disjoint union of \( m \) copies of \( G \). Terms are not defined here are used in the sense of Harary [1].

Mean labeling was introduced by S.Somasundaram and R. Ponraj [2] in 2003 and their behaviour studied in [2] and [3]. Harmonic mean labeling was introduced by S.Somasundaram, R.Ponraj and S.S. Sandhya [4] and their behaviour studied in [4] and [5]. Geometric mean labeling was introduced by S.Somasundram, P.Vidhyarani and R.Ponraj in [6]. In this paper we investigate geometric mean labeling of union of some graphs.
2. Geometric Mean Labeling

Definition 2.1

A graph $G = (V, E)$ with $p$ vertices and $q$ edges said to be a Geometric mean graphs if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $1, 2\ldots q+1$ in such a way that when edge $e = uv$ is labeled with $\lceil \sqrt{f(u)f(v)} \rceil$ (or) $\lfloor \sqrt{f(u)f(v)} \rfloor$ then the resulting edge labels are distinct. In this case $f$ is called a Geometric mean labeling of $G$.

Note 2.2

In a Geometric mean labeling of graphs, the vertices get labels from $1, 2\ldots q+1$ and edge from $1, 2\ldots q$.

$C_m$ and $P_n$ are geometric mean graphs [6].

Now we prove the following:

Theorem: 2.3

$C_m \cup P_n$ is a geometric mean graph for $m \geq 3$, $n \geq 1$.

Proof:

Let $C_m$ be the cycle $u_1u_2 \ldots u_mu_1$ and $P_n$ be the path $v_1v_2 \ldots v_n$.

Define a function

$f: V (C_m \cup P_n) \rightarrow \{1, 2\ldots q+1\}$ by

$f(u_i) = i \quad 1 \leq i \leq m$

$f(v_i) = m+i \quad 1 \leq i \leq n$

Then the set of labels of the edges of $C_m$ is $\{1, 2\ldots, m\}$. The set of labels of the edges of $P_n$ is $\{m+1, m+2\ldots m+n-1\}$.

Hence $C_m \cup P_n$ is a Geometric mean graph of $m \geq 3$ and $n \geq 1$. 
Example

Geometric Mean labeling of $C_5 \cup P_6$ is given below.

Next we have the following:

**Theorem 2.4**

$C_m \cup C_n$ is a geometric mean graph for $m \geq 3$, $n \geq 3$.

**Proof:**

Let $C_m$ be the cycle of the vertices $u_1, u_2, u_3 \ldots u_m$, $u_1$ and $C_n$ be the cycle of the vertices $v_1, v_2 \ldots v_n$, $v_1$.

Define a function

\[ f: V (C_m \cup C_n) \rightarrow \{1, 2 \ldots q+1\} \]

\[ f(u_i) = i, \quad 1 \leq i \leq m \]

\[ f(v_i) = m+i, \quad 1 \leq i \leq n \]

Then the set of labels of the edges of $C_m$ is $\{1, 2, 3 \ldots m\}$. The set of labels of the edges of $C_n$ is $\{m+1, m+2 \ldots m+n\}$.

Hence $C_m \cup C_n$ is a geometric mean graph.
Some results on geometric mean graphs

Example:

Geometric mean labeling of $C_5 \cup C_7$ is given below

Note: 2.5

$C_m \cup P_n$ and $C_m \cup C_n$ are mean graphs [2] as well as Harmonic mean graphs [5].

Theorem 2.6

$nK_3$ is a geometric mean graph for $n \geq 1$

Proof:

Let the vertex set of $nK_3$ be $V = V_1 \cup V_2 \ldots \cup V_n$ where $V_i = \{v^1_i, v^2_i, v^3_i\}$ and the edge set

$E = E_1 \cup E_2 \cup \ldots \cup E_n$ where $E_i = \{e^1_i, e^2_i, e^3_i\}$

Define

$f : V(nK_3) \rightarrow \{1, 2, 3 \ldots q +1\}$ by

$f(V^i) = 3(i-1)+j \quad 1 \leq i \leq n, 1 \leq j \leq 3.$

The set of labels of the edges of $nK_3$ is $\{1, 2, 3 \ldots 3n\}$.

Hence $nK_3$ is a geometric mean graph for $n \geq 1$
Example:

Geometric mean labeling of $3K_3$ is given below

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure3.png}
\caption{Figure 3}
\end{figure}

Now we have

**Theorem 2.7**

$nK_3 \cup P_m$ is a geometric mean graph for $n \geq 1$, $m \geq 1$.

**Proof:**

Let the vertex set of $nK_3$ be $U = U_1 \cup U_2 \cup U_3 .. \cup U_n$

Where $U_i = \{ u_{i1}, u_{i2}, u_{i3} \}$ and the edge set $E = E_1 \cup E_2 \cup E_3 \ldots \cup E_n$.

Where $E_i = \{ e_{i1}, e_{i2}, e_{i3} \}$

Let $P_m$ be the path $v_1 v_2 \ldots v_m$

Define a function

$$f: V (nK_3 \cup P_m) \rightarrow \{1, 2, 3 \ldots \ q + 1 \}$$ by

$$f (u_{ij}) = 3(i-1)+j \quad 1 \leq i \leq n$$

$$f (v_i) = 3n+i \quad 1 \leq j \leq m.$$  

The set of labels of the edges of $nK_3$ is $\{1, 2, 3 \ldots 3n\}$.

The set of label of the edges of $P_m$ is $\{3n+1, 3n+2, \ldots 3n+m-1\}$.

Hence $nK_3 \cup P_m$ is geometric mean graph for $n \geq 1$, $m \geq 1$.

**Example:**

Geometric mean labeling of $2K_3 \cup P_5$ is given below.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure4.png}
\caption{Figure 4}
\end{figure}
Theorem 2.8

\( nK_3 \cup C_m \) is a geometric mean graph for \( n \geq 1, m \geq 3 \).

**Proof:**

Let \( nK_3 \) be the \( n \) copies of \( K_3 \) graph. Let the vertex set of \( nK_3 \) be \( U = U_1 \cup U_2 \ldots \cup U_n \) where \( U_i = \{ u^1_i, u^2_i, u^3_i \} \) and the edge set \( E = E_1 \cup E_2 \cup \ldots \cup E_n \) where \( E_i = \{ e^1_i, e^2_i, e^3_i \} \). \( C_m \) be the cycle of \( v_1 v_2 \ldots v_n v_1 \)

Let us define a function

\[
 f : V( nK_3 \cup C_m ) \rightarrow \{ 1, 2, \ldots, q+1 \} 
\]

by \( f(u^j_i) = 3(i-1)+j \quad 1 \leq i \leq n, 1 \leq j \leq 3 \).

\( f(v_i) = 3n+i \quad 1 \leq i \leq m. \)

The set of label of the edges of \( nK_3 \) is \( \{ 1, 2, 3, \ldots, 3n \} \) and

The set of label of the edges of \( C_m \) is \( \{ 3n+1, 3n+2, \ldots, 3n+m \} \).

Hence \( nK_3 \cup C_m \) is geometric mean graph for \( n \geq 1, m \geq 3 \).

**Example:**

Geometric mean labeling of \( 2K_3 \cup C_6 \) is given below.

![Figure 5](image)

**Note 2.9**

\( nK_3, nK_3 \cup P_m, nK_3 \cup C_m \) are also Harmonic mean graphs [5].
Theorem 2.10

\( mC_n \) is a Geometric mean graph for \( m \geq 1, n \geq 3 \)

**Proof:**

Let the vertex set of \( mC_n \) be \( V = V_1 \cup V_2 \cup \ldots \cup V_m \)

Where \( V_i = \{ v_{i1}, v_{i2}, \ldots, v_{in} \} \) and the edge set \( E = E_1 \cup E_2 \cup \ldots \cup E_m \)

where \( E_i = \{ e_{i1}, e_{i2}, \ldots, e_{in} \} \) and

Define \( f: V(mC_n) \to \{1, 2, \ldots, q + 1\} \) by

\[
f(V_i) = n(i-1)+j \quad 1 \leq i \leq m, 1 \leq j \leq n.
\]

The set of labels of the edges of \( mC_n \) is \( \{1, 2, 3 \ldots, mn\} \)

Hence \( mC_n \) is geometric mean graph.

**Example:**

Geometric mean labeling of \( 3C_5 \) is given below:

![Figure 6]

Note: 2.11

\( mC_4 \) is also Harmonic mean graphs [5].

**Definition 2.12**

The square \( G^2 \) of a graph \( G \) has \( v(G^2) = V(G) \) with \( u, v \) adjacent in \( G^2 \) whenever \( d(u,v) \leq 2 \) in \( G \). The power of \( G^3, G^4 \ldots \) of \( G \) are similarly defined.
Theorem: 2.13

The graph $P_n^2$ is a geometric mean graph.

Proof:

Let $u_1, u_2, ..., u_n$ be the path $P_n$.

$P_n^2$ has $n$ vertices and $2n-3$ edges.

Define $f : V(P_n^2) \to \{1, 2, ..., q+1\}$ by $f(u_1) = 1$

$$f(u_i) = 2i-2, \quad 2 \leq i \leq n$$

The label of the edge $u_iu_{i+1}$ is $2i - 1 \quad (1 \leq i \leq n-1)$

The label of the edge $u_iu_{i+2}$ is $2i \quad (1 \leq i \leq n-2)$

Hence $P_n^2$ is geometric mean graph.

Example

Geometric mean labeling of $P_5^2$ is given below

![Figure 7](image_url)

Definition 2.14

The corona $G_1 \odot G_2$ of two graphs $G_1$ and $G_2$ is defined as the graph $G$ obtained by taking one copy of $G$ (which has $P$ vertices and $P$ copies of $G_2$) and then joining the $i$th vertex of $G_1$ to every vertices in the $i$th copy of $G_2$. Here we restrict ourselves to corona with cycles.
The graph $C_n \odot K_1$ is called a crown.

**Theorem 2.15:**

The crown $C_n \odot K_1$ is a geometric mean graph for all $n \geq 3$.

**Proof:**

Let $C_n$ be the cycle $u_1 u_2 \ldots u_n u_1$ and let $v_i$ be the vertex adjacent to $u_i \ (1 \leq i \leq n)$.

Define a function

$$f : V(C_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, q+1\}$$

by

- $f(u_i) = 2i-1$ for $1 \leq i \leq 2$.
- $f(u_i) = 2i+1$ for $3 \leq i \leq n$.
- $f(v_i) = 2i$ for $1 \leq i \leq n$.

Then the edge labels are distinct.

Hence crown is a geometric mean graph for all $n \geq 3$.

**Example:** Geometric mean labeling of $C_5 \odot K_1$ is given below
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