Bi-Distance Pattern Uniform Number

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Abstract

A graph $G = (V, E)$ is Bi-Distance Pattern Uniform if there exists $M \subseteq V(G)$ such that the $M$-distance pattern $f_M(u) = \{d(u, v) : v \in M\}$ is identical for all $u$ in $M$ and $f_M(v)$ is identical for all $v$ in $V - M$. The set $M$ is called Bi-DPU set. The least cardinality of Bi-DPU set in $G$ is called the Bi-DPU number of $G$. In this paper, we initiate a study on Bi-DPU number of different classes of graphs.

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1 Introduction

For all terminology and notation in graph theory, we refer the reader to Chartrand [1]. Unless mentioned otherwise, all graphs considered in this paper are finite, simple and connected.

Given an arbitrary non-empty subset $M$ of vertices in a graph $G = (V, E)$, each vertex $u$ in $G$ is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between the vertices $u$ and $v$ in $G$, is called the $M$-distance pattern of $u$ [3]. Germina and Rency [2] defined Bi-Distance Pattern Uniform (Bi-DPU) Graph as follows: Let $G = (V, E)$ be a $(p, q)$ graph...
and $M$ be any non-empty subset of $V(G)$. Then, the $M$-distance pattern of $u$ is the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between $u$ and $v$ in $G$. If $f_M(u)$ is identical for all $u \in M$ and $f_M(v)$ is identical for all $v \in V - M$, then $G$ is called a Bi-distance pattern uniform (Bi-DPU) graph. The set $M$ is called the Bi-DPU set. We need the following known results.

**Theorem 1.1.** [2] A non self-centered graph $G$ is a Bi-DPU graph if and only if $G$ has exactly two eccentricities and $\text{Cen}(G)$ is self-centered.

**Theorem 1.2.** [2] A tree $T$ is a Bi-DPU graph if and only if $T \cong K_{1,n}$ or $B_{m,n}$ where $B_{m,n}$ is a bistar.

### 1.1 Bi-DPU numbers of different classes of graphs

**Definition 1.3.** Bi-DPU number of a graph $G$, denoted by $\varsigma_B(G)$, is the minimum cardinality of a Bi-DPU set in $G$.

**Theorem 1.4.** A graph $G$ is a Bi-DPU graph with $\varsigma_B(G) = 1$ if and only if $G$ has at least one vertex of full degree.

**Proof.** Assume that $G$ has at least one full degree vertex. Let $u$ be a full degree vertex. Choose $M = \{u\}$. Then, $f_M(u) = \{0\}$ and $f_M(v) = \{1\}, \forall v \in V - M$. Hence, $\varsigma_B(G) = 1$. Conversely, let $\varsigma_B(G) = 1$, that is $|M| = 1$. Hence, $f_M(u) = \{0\}$ whenever $u \in M$ and $f_M(v) = \{1\}$ for all $v \neq u$, which implies the vertex in $M$ should necessarily be of full degree. \hfill $\square$

**Corollary 1.5.** $\varsigma_B(K_n) = 1$, $\varsigma_B(W_n) = 1$ where $W_n = K_1 + C_n$ is the Wheel graph and $\varsigma_B(K_{1,n}) = 1$.

**Corollary 1.6.** Given a fixed natural number $p$, for a $(p, q)$ graph with Bi-DPU number 1, we have, $p - 1 \leq q \leq \frac{p(p-1)}{2}$.

**Proof.** We have, $K_{1,p-1}$ and $K_p$ are Bi-DPU graphs with Bi-DPU number 1 of smallest and largest size respectively, we get $p - 1 \leq q \leq \frac{p(p-1)}{2}$. \hfill $\square$

**Theorem 1.7.** For bistar $B_{m,n}$, $\varsigma_B(B_{m,n}) = 2$.

**Proof.** Let $B_{m,n}$ be a bistar with Bi-DPU set $M = \{u, v\}$, where $u$ and $v$ are central vertices of $B_{m,n}$. That is, $\varsigma_B(B_{m,n}) \leq 2$. But, $B_{m,n}$ has no full degree vertex, $\varsigma_B(B_{m,n}) \neq 1$. Therefore, $\varsigma_B(B_{m,n}) = 2$. \hfill $\square$

**Theorem 1.8.** $\varsigma_B(K_{m,n}) = 2$, $m, n \geq 2$. 

Proof. Let \( \{X,Y\} \) be the bipartition of the vertex set of \( K_{m,n} \). Choose \( M = \{u,v\} \) where \( u \in X \) and \( v \in Y \). Then, \( f_M(u) = f_M(v) = \{0,1\} \) and \( f_M(w) = \{1,2\} \) \( \forall w \in V - M \). Hence, \( \zeta_B(K_{m,n}) \leq 2 \). Also, \( K_{m,n} \) contains no full degree vertex, \( \zeta_B(K_{m,n}) \neq 1 \). Hence, \( \zeta_B(K_{m,n}) = 2 \). \( \square \)

Theorem 1.9. \( \zeta_B(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \text{ is a multiple of 3} \\ \frac{n}{2}, & \text{if } n \text{ is even and not a multiple of 3} \\ n - 1, & \text{if } n \text{ is odd and not a multiple of 3} \end{cases} \)

Proof. Let \( C_n \) be a cycle on \( n \) vertices and \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \).

Case 1: \( n \) is a multiple of 3. Choose \( M = \{v_1, v_4, \ldots, v_{n-2}\} \). Then, for all \( v_i \in M \), \( f_M(v_i) = \{0,3,6,\ldots,\frac{n}{2}\} \) if \( n \) is even and \( \{0,3,6,\ldots,\frac{n}{2}-1\} \) if \( n \) is odd are identical sets.

Also, for all \( v_j \in V-M \), \( f_M(v_j) = \{1,2,4,5,7,8,\ldots,\frac{n-2}{2}\} \) if \( n \) is even and \( \{1,2,4,5,7,8,\ldots,\frac{n-1}{2}\} \) if \( n \) is odd are identical sets. Hence, \( \zeta_B(C_n) \leq \frac{n}{3} \) if \( n \) is a multiple of 3.

Now, we prove that \( \zeta_B(C_n) = \frac{n}{3} \). If possible, choose \( M' \subset V(C_n) \) with \( |M'| < |M| \). Let \( M' = \{u_1, u_2, \ldots, u_l\} \), where \( l < \frac{n}{3} \) and each \( u_j \) is some \( v_i \in V(C_n) \). Then, there exists at least one vertex \( u_j \in M' \) such that \( d(u_j, u_{j+1}) > 3 \). Assume \( d(u_j, u_{j+1}) = 4 \). Let the shortest \( u_j - u_{j+1} \)-path in \( C_n \) be \( u_j v_k v_{k+1} v_{k+2} u_{j+1} \). Then, \( 1 \in f_M(v_k) \) and \( 1 \notin f_M(v_{k+1}) \), \( M' \) can not be a Bi-DPU set for \( C_n \). Therefore, \( \zeta_B(C_n) = \frac{n}{3} \) whenever \( n \) is a multiple of 3.

Case 2: \( n \) is even and not a multiple of 3. Choose \( M = \{v_2, v_4, \ldots, v_n\} \), the set of all alternate vertices of \( C_n \).

Then, for all \( v_i \in M \),
\[
f_M(v_i) = \begin{cases} \{0,2,4,\ldots,\frac{n}{2}\}, & \text{if } n \text{ is even and } m \text{ is even} \\ \{0,2,4,\ldots,\frac{n-2}{2}\}, & \text{if } n \text{ is even and } m \text{ is odd} \end{cases}
\]

Therefore, \( f_M(v_i) \) are identical sets.

Also, for all \( v_j \in V-M \),
\[
f_M(v_j) = \begin{cases} \{1,3,5,\ldots,\frac{n-2}{2}\}, & \text{if } n \text{ is even and } m \text{ is even} \\ \{1,3,5,\ldots,\frac{n}{2}\}, & \text{if } n \text{ is even and } m \text{ is odd} \end{cases}
\]

Therefore, \( f_M(v_j) \) are identical sets. Hence, \( \zeta_B(C_n) \leq \frac{n}{2} \). Choose \( M' \subset V(C_n) \) with \( |M'| < \frac{n}{2} \) and if possible \( |M'| = \frac{n}{2} - 1 \). Let \( M' = \{u_1, u_2, \ldots, u_l\} \), where each \( u_j \) is some \( v_i \in V(C_n) \). Then, there exists at least one \( u_j \in M' \) such that \( d(u_j, u_{j+1}) \geq 3 \). In this case, let \( d(u_j, u_{j+1}) = 3 \), so that \( 3 \in f_{M'}(u_j) \), \( 3 \notin f_{M'}(u_{j-1}) \) and hence \( M' \) is not a Bi-DPU set. Now, when \( d(u_j, u_{j+1}) = 4 \), the shortest \( u_j - u_{j+1} \)-path in \( C_n \) is \( u_j v_k v_{k+1} v_{k+2} u_{j+1} \), so that \( 1 \notin f_{M'}(v_k) \), \( 1 \notin f_{M'}(v_{k+1}) \), \( M' \) is not a Bi-DPU set for \( C_n \). A similar argument follows when \( d(u_j, u_{j+1}) = 5, 6, \ldots, \frac{n}{2} \). Therefore, in all the cases \( |M'| \) is not a Bi-DPU set for \( C_n \). Hence, we conclude that \( \zeta_B(C_n) = \frac{n}{2} \) if \( n \) is even and not a multiple of 3.

Case 3: \( n \) is odd and not a multiple of 3. Choose \( M \) as the set of \( n-1 \)
vertices of $C_n$. Then, $f_M(v_i) = \{0, 1, 2, \ldots, \frac{n-1}{2}\}$, $\forall v_i \in M$ and $f_M(v_j) = \{1, 2, \ldots, \frac{n-1}{2}\}$, $\forall v_j \in V - M$. Hence, $\varsigma_B(C_n) \leq n - 1$. Choose $M' \subset V(C_n)$ with $|M'| < n - 1$. If $M'$ is a Bi-DPU set for $C_n$ then there are two possibilities. Either the elements of $M'$ are alternate vertices of $V(C_n)$ or there are two elements of $V - M'$ lies between the any two elements of $M'$. If the elements of $M'$ are alternate vertices of $V(C_n)$ then $|M'| = |V - M'|$, $|V(C_n)|$ is even, which is not possible. If there are two elements of $V - M'$ lies between the two elements of $|M'|$ then, $|V - M'| = 2|M'|$, $|V(C_n)|$ is a multiple of three, which is not possible. Hence, $M'$ is not a Bi-DPU set for $C_n$. Therefore, $\varsigma_B(C_n) = n - 1$ whenever $n$ is odd and not a multiple of 3.

The shadow graph $S(G)$ of a graph $G$ is obtained from $G$ by adding, for each vertex $v$ of $G$, a new vertex $v'$, called the shadow vertex of $v$, and joining $v'$ to the neighbors of $v$ in $G$.

**Theorem 1.10.** For the shadow graph $S(K_n)$ of complete graph, $\varsigma_B(S(K_n)) = n$.

**Proof.** Let the vertices of $K_n$ be $\{v_1, v_2, \ldots, v_n\}$ and the corresponding shadow vertices be $\{v_1', v_2', \ldots, v_n'\}$. Choose $M = \{v_1, v_2, \ldots, v_n\}$. Then, $f_M(v_i) = \{0, 1\}$, $\forall v_i \in M$ and $f_M(v_i') = \{1, 2\}$, $\forall v_i' \in V - M$. Hence, $\varsigma_B(S(K_n)) \leq n$. Choose $M' \subset V(S(K_n))$ such that $|M'| < |M|$.

**Case 1:** $M' \subset V(K_n)$

Then, there exists at least one $v_i \in V(K_n)$ which does not belong to $M'$ and for $v_i, v_i' \in V - M'$, $f_{M'}(v_i) = \{1\}$ and $f_{M'}(v_i') = \{1, 2\}$. Hence, $M'$ is not a Bi-DPU set.

**Case 2:** $M' \subset V(S(K_n)) - V(K_n)$

Then, there exists at least one $v_i' \in V(S(K_n)) - V(K_n)$ which does not belong to $M'$ and for $v_i, v_i' \in V - M'$, $f_{M'}(v_i) = \{1\}$ and $f_{M'}(v_i') = \{2\}$. Hence, $M'$ is not a Bi-DPU set.

**Case 3:** $M'$ consists of vertices of $K_n$, shadow vertices and $|M'| < n$.

Then, there exists $v_i, v_i' \in V - M'$ and since, $v_i$ is adjacent to all vertices of $V(S(K_n))$ except $v_i'$, $f_{M'}(v_i) = \{1\}$ and $2 \in f_{M'}(v_i')$. Hence, $M'$ is not a Bi-DPU set. Therefore, $\varsigma_B(S(K_n)) = n$.

**Theorem 1.11.** $\varsigma_B(P_m + P_n) = \begin{cases} 4 & \text{if } m, n \geq 4 \\ 1 & \text{otherwise.} \end{cases}$

**Proof.** Let $G \cong P_m + P_n$; $V(P_m) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$

**Case 1:** $m$ or $n < 4$

If $m$ or $n$ less than 4 then $G$ has at least one full degree vertex. Therefore, $\varsigma_B(P_m + P_n) = 1$.

**Case 2:** $m, n \geq 4$
Let $M = \{u_i, u_{i+1}, v_j, v_{j+1}\}, u_i \in V(P_n), v_j \in V(P_m)$. Then, $f_M(u) = \{0, 1\}$ for all $u \in M$ and $f_M(v) = \{1, 2\}$ for all $v \in V - M$. Then, $\varsigma_B(G) \leq 4$. Next, we prove that $\varsigma_B(G) \neq 4$. Since, $G$ has no full degree vertex, $\varsigma_B(G) \neq 1$. Also, $\varsigma_B(G) \neq 2$. For,

**Subcase 2.1.1:** Choose $M = \{u_i, v_j\}$ where $u_i \in V(P_n)$ and $v_j \in V(P_m)$. Then, $f_M(u_{i+1}) = f_M(u_{i-1}) = f_M(v_{j-1}) = f_M(v_{j+1}) = \{1\}$ and for all other vertices in $V - M$, $f_M(v) = \{1, 2\}$. Hence, $M$ is not a Bi-DPU set.

**Subcase 2.1.2:** Choose $M = \{u_i, u_j\}$. Then, $f_M(v_k) = \{1\}$ for all $v_k$, $f_M(u_i) = \{1, 2\}$ where $u_i$ is adjacent to $u_i$ or $u_j$ and $f_M(u_r) = \{2\}$ where $u_r$ is not adjacent to both $u_i$ and $u_j$. Hence, $M$ is not a Bi-DPU set. Now, $\varsigma_B(G) \neq 3$. For,

**Subcase 2.2.1:** Choose $M = \{u_i, u_j, v_k\}$. Then, $f_M(v_{k+1}) = f_M(v_{k-1}) = \{1\}$ and $f_M(v_s) = \{1, 2\}$ for all $v_s \in V - M$. Hence, $M$ is not a Bi-DPU set.

**Subcase 2.2.2:** Choose $M = \{u_i, u_j, v_k\}$. Then, $f_M(v_i) = \{1\}$ for all $v_i$ and $f_M(u_k) = \{1, 2\}$ for some $u_k \in V - M$. Hence, $M$ is not a Bi-DPU set. Therefore, we conclude that $\varsigma_B(P_m + P_n) = 4$. 

**Theorem 1.12.** The ladder $L_n \cong P_n \times P_2$ is a Bi-DPU graph if and only if $n \leq 4$ and $\varsigma_B(L_n) = \begin{cases} n & \text{if } n = 1, 2, 4 \\ 2 & \text{if } n = 3 \end{cases}$

**Proof.** First we prove that $L_n$ is a Bi-DPU graph for $n \leq 4$.

**Case 1:** When $n = 1$, $L_1 \cong K_2$, by theorem 1.4, $L_1$ is a Bi-DPU graph, $\varsigma_B(L_1) = 1$.

**Case 2:** When $n = 2$, $L_2 \cong C_4$, by theorem 1.9, $L_2$ is a Bi-DPU graph, $\varsigma_B(L_2) = 2$.

**Case 3:** $n = 3$. Let $v_1, v_2, v_3$ and $v_4$ be the vertices of $L_3$ corresponding to the eccentricity 3 and $u_1$ and $u_2$ be the vertices of $L_3$ corresponding to the eccentricity 2. Choose $M = \{u_1, u_2\}$. Then, $f_M(u_1) = f_M(u_2) = \{0, 1\}$ and $f_M(v) = \{1, 2\}$ for all $v \in V - M$. Hence, $M$ is a Bi-DPU set for $L_3$ and $\varsigma_B(L_3) \leq 2$. Since, $L_3$ has no full degree vertex, $\varsigma_B(L_3) \neq 1$. Therefore, $\varsigma_B(L_3) = 2$.

**Case 4:** $n = 4$. Let $v_1, v_2, v_3$ and $v_4$ be the vertices of $L_4$ corresponding to the eccentricity 4 and $u_1, u_2, u_3$ and $u_4$ be the vertices of $L_4$ corresponding to the eccentricity 3. Choose $M = \{u_1, u_2, u_3, u_4\}$. Then, $f_M(u_i) = \{0, 1, 2\}$ for $i = 1, 2, 3, 4$ and $f_M(v_j) = \{1, 2, 3\}$ for $j = 1, 2, 3, 4$. Therefore, $M$ is a Bi-DPU set for $L_4$ and $\varsigma_B(L_4) \leq 4$. We prove $\varsigma_B(L_4) \neq 4$. Since, $L_4$ contains no full degree vertex, $\varsigma_B(L_4) \neq 1$. Also, $\varsigma_B(L_4) \neq 2$. For,

**Subcase 4.1.1:** Choose $M = \{v_j, u_i\}$. Then, there exists $v_k \in V - M$ such that $4 \in f_M(v_k)$ and $4 \notin f_M(v)$ for all $v \in (V - M) - \{v_k\}$. Hence, $M$ is not a Bi-DPU set for $L_4$.

**Subcase 4.1.2:** Choose $M = \{v_i, v_j\}$. If $d(v_i, v_j) = 1$ or 3 then $4 \in f_M(v_k)$ for all $v_k \in V - M$ and $4 \notin f_M(u_i)$ for all $u_i \in V - M$. If $d(v_i, v_j) = 4$ then
\[ f_M(u_1) = f_M(u_3) = \{1, 3\} \text{ and } f_M(u_2) = f_M(u_4) = \{2\}. \] Hence, \( M \) is not a Bi-DPU set for \( L_4 \).

**Subcase 4.1.3:** Choose \( M = \{u_i, u_j\} \). Then, there are two vertices \( v_l, v_k \in V - M \) such that \( 3 \in f_M(v_l), f_M(v_k) \) and \( 3 \not\in f_M(v) \) for all \( v \in (V - M) - \{v_l, v_k\} \). Hence, \( M \) is not a Bi-DPU set for \( L_4 \).

Now, \( \varsigma_B(L_4) \neq 3 \). For,

**Subcase 4.2.1:** Choose \( M = \{u_i, u_j, v_k\} \). If any two vertices in \( M \) are adjacent then \( f_M(u) \) is not identical for all \( u \in M \). If all the vertices in \( M \) are non-adjacent then there exists a vertex \( v_s \in V - M \) such that \( 4 \in f_M(v_s) \) and \( 4 \not\in f_M(v) \) for all \( v \in (V - M) - \{v_s\} \). Hence, \( M \) is not a Bi-DPU set for \( L_4 \).

**Subcase 4.2.2:** Choose \( M = \{v_i, v_j, u_k\} \). If \( v_i \) is adjacent to both \( u_j \) and \( v_k \) then \( f_M(v_i) = \{0, 1\} \) and \( f_M(v_j) = f_M(u_k) = \{0, 1, 2\} \). If any two vertices in \( M \) are adjacent then \( f_M(v_i) \neq f_M(v_j) \neq f_M(u_k) \). If no two elements in \( M \) are adjacent and \( d(v_i, v_j) = 3 \) then \( 4 \in f_M(v_s), f_M(v_k) \) and \( 4 \not\in f_M(v) \) for all \( v \in (V - M) - \{v_s, v_k\} \). If no two elements in \( M \) are adjacent and \( d(v_i, v_j) = 4 \) then \( 4 \in f_M(v_i), f_M(v_j) \) and \( 4 \not\in f_M(u_k) \). Hence, \( M \) is not a Bi-DPU set for \( L_4 \).

Therefore, we conclude that \( \varsigma_B(L_4) = 4 \).

Conversely, assume that \( L_n \) is a Bi-DPU graph. We have to prove that \( L_n \) is a Bi-DPU graph only for \( n \leq 4 \). If possible suppose \( n \geq 5 \). Then, \( L_n \) has more than two eccentricities. Hence, by theorem 1.4, \( L_n \) is not a Bi-DPU graph. Therefore, \( L_n \) is a Bi-DPU graph for \( n \leq 4 \). \( \square \)

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