International Mathematical Forum, Vol. 7, 2012, no. 24, 1179 - 1186

A Meshfree Approach for the Numerical Solution of Nonlinear sine-Gordon Equation

Marjan Uddin [1]

BSI, UET Peshawar, KPK Pakistan

Sirajul Haq

GIK Institute Topi, KPK, Pakistan

Ghulam Qasim

BSI, UET Peshawar, KPK Pakistan

Abstract

A numerical method based on radial basis functions (RBFs) is proposed for the numerical solution of nonlinear sine-Gordon equation. The numerical method is based on scattered data interpolation along with basis functions known as radial basis functions (RBFs). The spatial derivatives are approximated by the derivatives of interpolation and a low order scheme is used to approximate temporal derivative. The scheme is tested for single soliton, collision of breathers and soliton doublets. The results obtained from the method are compared with the exact solutions and the earlier work.

Mathematics Subject Classification: 65M20

Keywords: RBFs, sine-Gordon equation, RBFs interpolation method

1 Introduction

In the last two decades, the study of solutions of nonlinear waves equations has gained much attention. These waves equations has applications in many branches of applied mathematics and theoretical physics. Zabusky and Kruskal [1] observed that KdV equation leads to solitary waves called soliton, is a localized entity which keeps its identity after interaction. The sine-Gordon

[1] marjankhan1@hotmail.com (corresponding author)

equation also model a soliton wave, which occurs in several physical situation [2].

In the last two decade, RBFs approximation method has been considered as a power technique for solving differential equations. Kansa [4] derived a multiquadric scheme for the numerical solution of PDEs. Recently various ordinary and partial differential equations have been solved using this method [5, 6]. The existence, uniqueness, and convergence of the RBFs approximation was discussed by Micchelli [7], and Schaback [8]. RBFs approximation method has achieved very high accuracy in solving partial differential equation with very easy implementation. This method has an edge over finite element, finite difference and finite volume methods, because it does not require a mesh in problem domain. Derivatives of the RBFs are used to approximate spatial derivatives using a set of nodes scattered in the problem domain. However the interpolation matrix obtained in this technique becomes highly ill-conditioned when the number of nodes increases [9]. Several techniques have been used to overcome this drawback, (see the references [10]).
In this work, we apply RBFs approximation method using the multiquadric radial basis function (MQ) ($(r^2+c^2)^{1/2}$ where c is a shape parameter), for the numerical solution of sine-Gordon equation given by

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sin u, \quad (x,t) \in [a,b] \times [0,T], \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u_1(x), \; u_t(x,0) = u_2(x), \; u(a,t) = g_1(t), \; u(b,t) = g_2(t). \tag{2}$$

The structure of the present paper is organized as follows. In Section 2, we discuss the meshfree method for sine-Gordon equation. Section 3, is devoted to the numerical tests of the method on the problems related to the sine-Gordon equation. In Section 4, the results are concluded.

2 Analysis of the method for sine-Gordon equation

The transformed form of sine-Gordon equation (1), is given as

$$u_t(x,t) = v(x,t), \quad v_t(x,t) = u_{xx}(x,t) - \sin u(x,t), \tag{3}$$

the boundary conditions are

$$u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \quad v(a,t) = g_1(t), \quad v(b,t) = g_2(t), \quad t > 0, \tag{4}$$

and initial conditions

$$u(x,0)=f(x),\quad v(x,0)=g(x)\quad a\leq x\leq b. \tag{5}$$

We use the following scheme for equation (3)

$$\frac{1}{\delta t}(U^{n+1}-U^{n})=V^{n},\ \ \frac{1}{\delta t}(V^{n+1}-V^{n})=(U_{xx})^{n}-\sin U^{n}, \tag{6}$$

rearranging equation (6), we get

$$U^{n+1}=U^{n}+\delta tV^{n},\ \ V^{n+1}=V^{n}+\delta t\left((U_{xx})^{n}-\sin U^{n}\right) \tag{7}$$

where $t^{n+1}=t^{n}+\delta t$. The RBFs approximations for the solutions $u(x,t)$ and $v(x,t)$ of equation (3) are given by

$$U^{n}(x_i)=\sum_{j=1}^{N}\lambda_{1j}^{n}\phi(r_{ij}),\quad V^{n}(x_i)=\sum_{j=1}^{N}\lambda_{2j}^{n}\phi(r_{ij}),\quad ,i=1,...,N. \tag{8}$$

Using (8) along with the boundary conditions (4), the system of equations in (7) can be written in matrix form

$$\mathbf{A}\lambda_{\mathbf{1}}{}^{n+1}=\mathbf{A}\lambda_{\mathbf{1}}{}^{n}+\delta t\mathbf{V^{n}}+\mathbf{f}^{n+1},\quad \mathbf{A}\lambda_{\mathbf{2}}{}^{n+1}=\mathbf{A}\lambda_{\mathbf{2}}{}^{n}+\delta t\left(\mathbf{U_{xx}^{n}}-\sin\mathbf{U^{n}}\right)+\mathbf{g}^{n+1}, \tag{9}$$

where $\mathbf{A}=[\phi(r_{ij})]_{i,j=1}^{N}$. In more compact we have

$$\lambda_{\mathbf{1}}{}^{n+1}=\mathbf{I}\lambda_{\mathbf{1}}{}^{n}+\mathbf{A^{-1}}\mathbf{F}^{n+1},\ \lambda_{\mathbf{2}}{}^{n+1}=\mathbf{I}\lambda_{\mathbf{2}}{}^{n}+\mathbf{A^{-1}}\mathbf{G}^{n+1}. \tag{10}$$

Where
$\mathbf{f^{n+1}}=\left[\ f_1^{n+1},0,0,...,f_2^{n+1}\ \right]^T,\quad \mathbf{g^{n+1}}=\left[\ g_1^{n+1},0,0,...,g_2^{n+1}\ \right]^T,$
$\mathbf{F^{n+1}}=\left[\mathbf{f^{n+1}}+\delta t\mathbf{V^{n}}\right],\quad \mathbf{G^{n+1}}=\left[\mathbf{g^{n+1}}+\delta t\left(\mathbf{U_{xx}^{n}}-\sin\mathbf{U^{n}}\right)\right].$
Equations in (8) can be written in matrix form as

$$\mathbf{U}^{n}=\mathbf{A}\lambda_{\mathbf{1}}{}^{n},\quad \mathbf{V}^{n}=\mathbf{A}\lambda_{\mathbf{2}}{}^{n}. \tag{11}$$

Using equation (11) in equation (10), we get

$$\mathbf{U}^{n+1}=\mathbf{U}^{n}+\mathbf{F^{n+1}},\ \mathbf{V}^{n+1}=\mathbf{V}^{n}+\mathbf{G^{n+1}}. \tag{12}$$

From here we can find the solution at any time level n.

3 Numerical examples

In this section, we apply the proposed method for the numerical solution of sine-Gordan equation. The accuracy of the meshfree method is tested in terms of L_2, L_∞ error norms and the conservation of energy of sine-Gordon equation. These error norms and energy are defined as

$$L_2 = \|u-U\|_2 = \left(\delta x \sum_{j=1}^{N} (u-U)^2\right)^{1/2}, \quad L_\infty = \|u-U\|_\infty = \mathrm{m}ax_j \big|u-U\big|. \tag{13}$$

$$E = 1/2 \int_{-\infty}^{\infty} \left[(u_t)^2 + (u_x)^2 + 2(1-\cos u)\right] \, dx, \tag{14}$$

where U and u denote the numerical and exact solution respectively. The test problems are given below.

Problem 1. Single soliton:
We consider sine-Gordon equation (1) as system of two equations

$$u_t = v, \quad v_t = u_{xx} - \sin u, \tag{15}$$

the exact solutions of the equations in (15) are given as

$$u(x,t) = 4\tan^{-1}\left(\exp\left[\gamma(x-Ct)+\beta\right]\right), \; v(x,t) = \frac{-4\gamma C \exp\left[\gamma(x-Ct)+\beta\right]}{1+\exp\left[\gamma(x-Ct)+\beta\right]^2}. \tag{16}$$

Where $\gamma = (1-C^2)^{-1/2}$. The initial conditions $u(x,0)$, $v(x,0)$ and boundary conditions $u(a,t)$, $u(b,t)$, $v(a,t)$, $v(b,t)$ are used from the exact solutions (16). We solved the problem in the spatial interval $-2 \le x \le 58$. The results are displayed in Table 1 and Figure 1. It is observed that the energy $E(t)$ is almost constant which confirm the accuracy of the meshfree method. The results has good agreement with exact solution and better than earlier work [3]. It is observed that the solution converges when $0.2 < c < 4.6$, and the solution accuracy increases and with the decrease in time step size δt.

Problem 2. Soliton-antisoliton breather:
We consider Sine-Gordon equation (1) as system of two equations

$$u_t = v, \quad v_t = u_{xx} - \sin u, \tag{17}$$

the exact solutions of a breather (or bion), which is a bound kink-antikink pair and the energy, are given by

$$u(x,t) = 4\tan^{-1}\left[C^{-1}\sin(\bar{\gamma}Ct)\mathrm{sech}(\bar{\gamma}x)\right], \; v(x,t) = \frac{4\bar{\gamma}\cos(\bar{\gamma}Ct)\mathrm{sech}(\bar{\gamma}x)}{1+\left[C^{-1}\sin(\bar{\gamma}Ct)\mathrm{sech}(\bar{\gamma}x)\right]^2}. \tag{18}$$

Table: 1
Error norms and energy constant when single soliton for $\delta t = 0.001$, $N = 121$, $c = 1$, $C = 0.5$, $\beta = 0$ in $[-2, 58]$ corresponding to problem 1.

Time	L_∞	L_2	[3] L_∞	$E(t)$
9	6.836E-003	2.683E-002	4.518E-001	9.241144
36	8.032E-002	3.113E-001	2.293E+000	9.248183
108	6.253E-001	2.378E+000	5.120E+000	9.267047

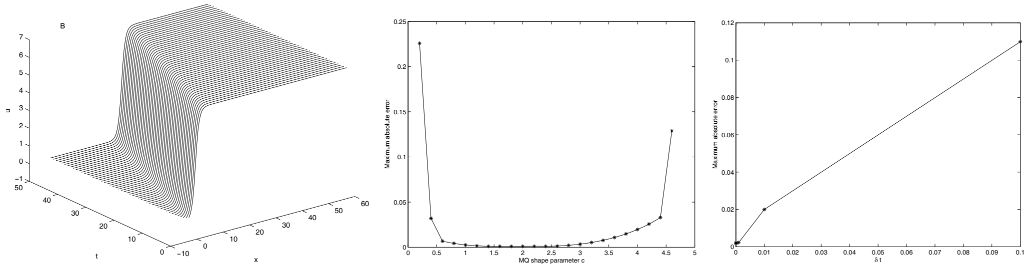

Figure 1: Single soliton: The numerical solution over the time interval [1,50], MQ shape parameter c versus L_∞ and δt versus L_∞ are shown, when $C = 0.5, \beta = 0, \delta t = 0.001, N = 121$. corresponding to problem 1.

Where $\bar{\gamma} = (1 + C^2)^{-1/2}$, $E = 16\bar{\gamma}$. This problem is solved in the spatial interval $[-10, 10]$. We display the numerical solution in Table 2 and Figure 2. The results has a good agreement with exact solution and the earlier work by [3].

Problem 3. Soliton-soliton doublets:
We consider Sine-Gordon equation (1) as system of two equations

$$u_t = v, \qquad v_t = u_{xx} - \sin u, \tag{19}$$

Table: 2
Error norms and energy constant when single soliton for $\delta t = 0.001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-10, 10]$ corresponding to problem 2.

Time	L_∞	L_2	[3] L_∞	$E(t)$
1	1.474E-003	1.252E-002	0.988E-003	14.320090
10	9.215E-003	9.402E-002	0.1622E-002	14.342018
20	3.038E-001	4.599E+000	0.103E-002	14.364043

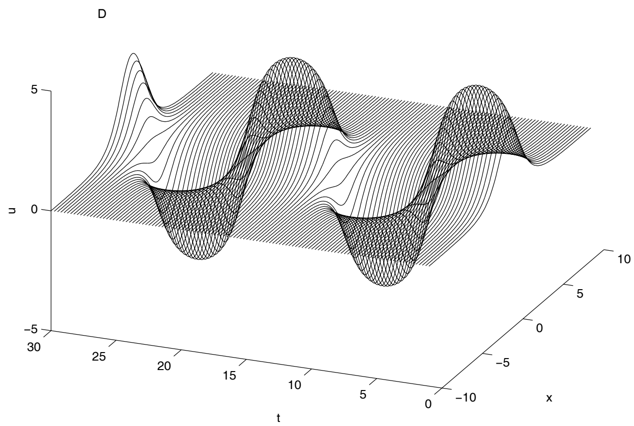

Figure 2: Soliton-antisoliton breather: The numerical solution is shown over the time interval [1,20] for $C = 0.5, \beta = 0, \delta t = 0.001, N = 201$. corresponding to problem 2.

Table: 3
Error norms and energy constant when single soliton for $\delta t = 0.001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-20, 20]$ corresponding to problem 3.

Time	L_∞	L_2	[3] L_∞	$E(t)$
2	1.568E-003	3.025E-011	0.127E-003	18.476461
10	3.151E-003	3.695E-012	0.191E-003	18.480671
20	1.828E-002	1.039E-009	0.251E-003	18.485837

the exact solutions and the energy, are given by

$$u(x,t) = 4\tan^{-1}\left[C\text{sech}(\gamma C t)\sinh(\gamma x)\right], \tag{20}$$

$$v(x,t) = \frac{-4C^2\gamma\text{sech}(\gamma C t)\tanh(\gamma C t)\sinh(\gamma x)}{1 + \left[C\text{sech}(\gamma C t)\sinh(\gamma x)\right]^2}.$$

Where $\gamma = (1 - C^2)^{-1/2}$, $E = 16\gamma$. The initial conditions $u(x,0), v(x,0)$ and boundary conditions $u(a,t)$, $u(b,t)$, $v(a,t)$, $v(b,t)$ can be obtain from the exact solutions in (20). In Table 3 and Figure 3, we displayed the numerical solution over the time interval $[1, 30]$. It is noted that the results of the present method has a good agreement with exact solution and the earlier work by [3] as shown in Table 3 and Figure 3.

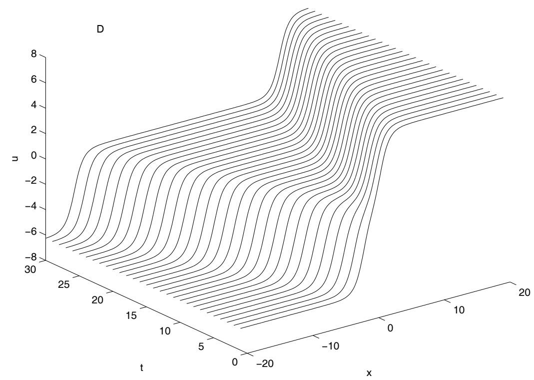

Figure 3: Soliton-soliton doublets: The numerical solution is shown over the time interval [1,30] when $C = 0.5, \beta = 0, \delta t = 0.001, N = 201$. corresponding to problem 3.

4 Concluding remarks

In this paper, a meshfree interpolation method using radial basis functions is applied for the numerical solution of sine-Gordon equation. The results of this method has a good agreement with exact solution and with earlier work [3]. The technique used in this paper provides an efficient alternative for the solution of nonlinear partial differential equations.

References

[1] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15** (1965), 240-243.

[2] P. J. Caudrey, J. C. Eilbeck, and J. D. Gibbon, The sine-Gordon equation as a model classical field theory, *Nuovo Cimento* **25** (1975) 497-511.

[3] A. G. Bratsos, A fourth order numerical scheme for the one-dimensional sine-Gordon equation, *Int.J. Comput. Math.* **85** (2008) 1083-1095.

[4] E. J. Kansa, *Multiquadrics scattered data approximation scheme with applications to computational fluid-dynamics-I, surface approximations and partial derivative estimates, Comput. Math. Appl.* **19** (1990) 127-145.

[5] Sirajul Haq, Marjan Uddin RBFs approximation method for Kawahara equation, *Appl. Math. Comput.,* **35** (2011) 575-580.

[6] Marjan Uddin, Sirajul Haq, *RBFs approximation method for time fractional partial differential equations, Commun. Nonlin. Sci.Numer. Simu.,* **16** (2011) 4208-4214.

[7] C. A. Micchelli, *Interpolation of scatterted data: distance matrix and conditionally positive definite functions Construct. Approx.*, **2** (1986) 11-22.

[8] C. Franke, R. Schaback, *Convergence order estimates of meshless collocation methods using radial basis functions*, *Adv. Comput. Math.* **8** (1998) 381-99.

[9] R. Schaback, Error estimates and condition numbers for radial basis function interpolation. *Adv. Comput. Math.* **3** (1995) 251-264.

[10] J. C. Li, Y. C. Hon, Domain decomposition for radial basis meshless methods. *Numer. Methods Part. Differ. Eq.* **20** (2004) 450-462.

Received: November, 2011