The L-Dual of a Generalized m-Kropina Space of Order Two

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Abstract

In ([8]), ([5]) the prolongation to $\text{Osc}^2 M$ of Riemannian, Finslerian and Lagrangian structures were introduced. The L-dual of a generalized m-Kropina space was introduced in [12]. In this paper we give the L-dual of a generalized m-Kropina space of second order using Lagendre transformation of second order.

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1 Introduction

The L-duality of Finsler and Lagrange spaces was introduced by R. Miron ([7]) and was intensively studied by others, including the first author of this article.

Concrete cases of Hamiltonians obtained by L-duality methods were also constructed. In special the L-duals of some $(\alpha, \beta)$-metrics like Randers and Kropina are quite interesting ([2], [3]). The L-dual of another famous $(\alpha, \beta)$-metric, namely the Matsumoto metric was introduced in [9]. In [10], the L-dual of an
(α, β) Finsler space of order two was introduced by I. M. Masca and others. Moreover, very recently, the L-dual of a generalized m-Kropina space was introduced in [12].

A natural questions arises: what is the L-dual of a generalized m-Kropina space of second order. In this paper this is the question we are going to answer.

2 The Legendre transformation

Let us consider a Lagrange space of order two ([5]) denoted by $L(2)^n = (M, L(x, y^{(1)}, y^{(2)}))$, where $L : (x, y^{(1)}, y^{(2)}) \in T^2 M \rightarrow L(x, y^{(1)}, y^{(2)}) \in R$ is the fundamental function and $c_{ij}$ is the fundamental metric tensor given by:

$$c_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(2)i} \partial y^{(2)j}}.$$  \hspace{1cm} (1)

If M is paracompact manifold, the existence of second order Lagrange spaces, with positively defined fundamental tensor field is always assured ([8]). In this case, there is also a Riemannian metric a on M. Then, the Liouville d-vector $z^{(2)i} = y^{(2)i} + \frac{1}{2} \gamma^i_{jk} y^{(1)j} y^{(1)k}$, \hspace{1cm} (2)

is globally defined on $\tilde{E}$, where

$$\tilde{E} = Osc^2 M \setminus \{0\} = \{ (x, y^{(1)}, y^{(2)}) \in Osc^2 M | \text{rank}||y^{(1)i}|| = 1 \}$$ \hspace{1cm} (3)

and it depends only on the metric a. Here $\gamma^i_{jk}$ are the Christoffel symbols of Riemannian metric a.

The Liouville d-vector $z^{(2)i}$ allows us to construct not only the regular Lagrangian:

$$L(x, y^{(1)}, y^{(2)}) = \frac{1}{2} (a_{ij}(x) z^{(2)i} z^{(2)j})^2,$$ \hspace{1cm} (4)

but also some others, for example, putting $\alpha^2 = a_{ij}(x) z^{(2)i} z^{(2)j}$ and $\beta = b_i(x) z^{(2)i}$ a differential linear function in $z^{(2)i}$. This is the Prolongation of a Riemannian metric to $Osc^2 M$, introduced by R. Miron in ([7]).

It is known, ([6]), a Finsler space of order two $F^{(2)n} = (M, F(x, y^{(1)}, y^{(2)}))$ is a Lagrange space of second order $L(2)^n = (M, L(x, y^{(1)}, y^{(2)}))$ with

$$L(x, y^{(1)}, y^{(2)}) = F^2(x, y^{(1)}, y^{(2)}),$$ \hspace{1cm} (5)

having the fundamental function $F$ positively, 2-homogeneous with respect to $y^{(2)i}$, the fundamental tensor $c_{ij}$ positively defined. In this way, we can define
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an \((\alpha, \beta)\) Finsler space of order two as follows:

1. A Randers space of second order having the fundamental function:

\[ F(x, y^{(1)}, y^{(2)}) = \alpha(x, y^{(1)}, y^{(2)}) + \beta(x, y^{(1)}, y^{(2)}), \]

\[ F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\beta(x, y^{(1)}, y^{(2)})} \]

2. A Kropina space of order two with fundamental function:

\[ F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\alpha(x, y^{(1)}, y^{(2)}) - \beta(x, y^{(1)}, y^{(2)})} \]

3. A Matsumoto space of order two with fundamental function:

\[ F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^{m+1}(x, y^{(1)}, y^{(2)})}{\beta^m(x, y^{(1)}, y^{(2)})} \]

4. A Generalized m-Kropina space of order two with:

\[ F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^{m+1}(x, y^{(1)}, y^{(2)})}{\beta^m(x, y^{(1)}, y^{(2)})} \]

The fundamental function is called, like in classical case, an \((\alpha, \beta)\)-metric if \(F\) is homogeneous of \(\alpha\) and \(\beta\) of degree two.

Let us consider a Hamilton space of order two \(H^{(2)}n = (M, H(x, y, p))\) with the regular Hamiltonian \(H : T^2M \to R\), differentiable on \(T^{*2}M\) and continuous on the zero section of the projection \(\pi^{*2} : T^{*2}M \to M\), having the fundamental tensor field:

\[ g^{ij}(x, y, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}, \]

with constant signature on the manifold \(T^{*2}M\).

Let \(C^{(2)}n = (M, K(x, y, p))\) be a Cartan space of order two. From ([8]) it is known that it is a Hamilton space of second order \(H^{(2)}n\) for which the fundamental function \(H(x, y, p)\) is 2-homogeneous with respect to momenta \(p_i\) and

\[ H(x, y, p) = K^2(x, y, p). \]

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In this case, we put: \(\alpha^2 = y_iy^i, \ \beta = b_iy^i, \ \beta^* = b_ip^i, \ p^i = a^{ij}p_j, \ \alpha^2 = p_ip^i = a^{ij}p_ip_j\). We have \(F = \frac{\alpha^{m+1}}{\beta^m}\), and

\[ p_i = \frac{1}{2} \frac{\partial F^2}{\partial y^{(2)i}} = F\dot{\alpha}\left(\frac{\alpha^{m+1}}{\beta^m}\right) \]
which on simplification gives

\[ p_i = F \left[ \frac{(m+1)F}{\alpha^2} a_{ij}z^{(2)j} - \frac{mF}{\beta} b_i \right]. \]  (12)

Contracting in (12) by \( p^i \) and \( b^i \) respectively, we get

\[ \alpha^{*2} = F \left[ (m+1)F \left( \frac{\alpha^{2m}}{\beta^{2m}} \right) - m\beta^{*} \left( \frac{\alpha^{m+1}}{\beta^{m+1}} \right) \right] \]  (13)

and

\[ \beta^{*} = F \left[ (m+1)\frac{\alpha^{m-1}}{\beta^{m-1}} - mb^{2} \frac{\alpha^{m+1}}{\beta^{m+1}} \right] \]  (14)

A generalized m-Kropina space of order two is a special \((\alpha, \beta)\)-metric with \( \phi = \frac{1}{s^m} \).

Using Shen’s [13] notation \( s = \frac{\beta}{\alpha} \) above equation becomes :

\[ \alpha^{*2} = F \left[ \frac{(m+1)F}{s^{2m}} - \frac{m\beta^{*}}{s^{m+1}} \right] \]  (15)

and

\[ \beta^{*} = F \left[ \frac{(m+1)}{s^{m-1}} - \frac{mb^{2}}{s^{m+1}} \right] \]  (16)

Putting \( s^{m} = t \), so that \( s = t^{\frac{1}{m}} \) in (15) and (16), we get

\[ \alpha^{*2} = \frac{(m+1)F^2}{t^2} - \frac{mF\beta^{*}}{t^{m+1}} \]  (17)

and

\[ \beta^{*} = \frac{(m+1)F}{t^{\frac{m-1}{m}}} - \frac{mb^{2}}{t^{\frac{m+1}{m}}} \]  (18)

From (18), we get

\[ \beta^{*}t^2 = F \left[ (m+1)t^{\frac{m+1}{m}} - mb^{2}t^{\frac{m-1}{m}} \right] \]  (19)

For \( b^2 = 1 \), from (19), we get

\[ F = \frac{\beta^{*}t}{(m+1)t^{\frac{m}{m}} - mt^{\frac{1}{m}}} \]  (20)
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put the value of $F$ in (17), we get

$$\alpha^{*2}(m + 1)^2 t^\frac{4}{m} - \{2m(m + 1)\alpha^{*2} - (m^2 - 1)\beta^{*2}\} t^\frac{m}{m} + m^2(\alpha^{*2} - \beta^{*2}) = 0$$

or

$$\alpha^{*2}(m + 1)^2 s^4 - \{2m(m + 1)\alpha^{*2} - (m^2 - 1)\beta^{*2}\} s^2 + m^2(\alpha^{*2} - \beta^{*2}) = 0$$

(21)

solving (21), we get

$$s^2 = \frac{1}{2} \frac{1}{\alpha^{*2}(m + 1)} \left\{ (2m\alpha^{*2} - (m - 1)\beta^{*2}) \pm \beta^* \sqrt{4m\alpha^{*2} + (m - 1)^2}\beta^{*2} \right\}$$

$$\Rightarrow s = \begin{cases} \sqrt{2m\alpha^{*2} - (m - 1)\beta^{*2} \pm \beta^* \sqrt{4m\alpha^{*2} + (m - 1)^2}\beta^{*2}} \ci{2(m + 1)\alpha^{*2}} \end{cases}$$

Hence, we get

$$s = \sqrt{\frac{c \pm \beta^* \sqrt{d}}{e}}$$

(22)

where

$$c = 2m\alpha^{*2} - (m - 1)\beta^{*2},$$

$$d = 4m\alpha^{*2} + (m - 1)^2\beta^{*2},$$

$$e = 2(m + 1)\alpha^{*2}$$

Using (22) in (20), we get

$$F = \frac{\beta^* \left( \frac{c \pm \beta^* \sqrt{d}}{e} \right)^{m+1}}{(m + 1) \left( \frac{c \pm \beta^* \sqrt{d}}{e} \right) - m}$$

(23)

Hence $H(x, p) = \frac{1}{2}F^2$ is given by

$$H(x, p) = \frac{1}{2} \frac{\beta^{*2} \left( \frac{c \pm \beta^* \sqrt{d}}{e} \right)^{m+1}}{(m + 1) \left( \frac{c \pm \beta^* \sqrt{d}}{e} \right) - m}$$

(24)

Putting $\beta^* = b^i p_i$, $\alpha^* = (a^{ij}(x)p_ip_j)^{\frac{1}{2}}$ in (24), we get

$$H(x, p) = \frac{1}{2} \frac{(b^i p_i)^2 \left( \frac{c \pm (b^i p_i)\sqrt{d}}{e} \right)^{m+1}}{(m + 1) \left( \frac{c \pm (b^i p_i)\sqrt{d}}{e} \right) - m}$$

(25)
Next, we find \( H(x, p) \) for \( b^2 \neq 1 \). From (19), we have

\[
F = \frac{\beta^* t}{(m + 1) t^{\frac{m}{m+1}} - mb^2 t^{-\frac{1}{m+1}}} \tag{26}
\]

Using (26) in (17), we get

\[
\alpha^2 (m + 1)^2 t^\frac{4}{m} - \left\{ 2m(m + 1)\alpha^2 b^2 - (m^2 - 1)\beta^* \right\} t^\frac{2}{m} + m^2(\alpha^2 b^2 - \beta^*) = 0
\]

or

\[
\alpha^2 (m + 1)^2 s^4 - \left\{ 2m(m + 1)b^2 \alpha^2 - (m^2 - 1)\beta^* \right\} s^2 + m^2(\alpha^2 b^2 - \beta^*) = 0 \tag{27}
\]

solving (27), we get

\[
s^2 = \frac{1}{2} \frac{1}{\alpha^2 (m + 1)} \left\{ 2m\alpha^2 b^2 - (m - 1)\beta^* \pm \beta^* \sqrt{4mb^2\alpha^2 + (m - 1)^2\beta^*} \right\}
\]

\[
\Rightarrow s = \left[ \sqrt{\frac{2m\alpha^2 b^2 - (m - 1)\beta^* \pm \beta^* \sqrt{4mb^2\alpha^2 + (m - 1)^2\beta^*}}{2(m + 1)\alpha^2}} \right]
\]

Hence, we get

\[
s = \sqrt{\frac{f \pm \beta^* \sqrt{g}}{e}}, \tag{28}
\]

where

\[
f = 2mb^2\alpha^2 - (m - 1)\beta^*,
\]

\[
g = 4mb^2\alpha^2 + (m - 1)^2\beta^*,
\]

\[
e = 2(m + 1)\alpha^2.
\]

Using (28) in (26), we get

\[
F = \beta^* \left( \frac{f \pm \beta^* \sqrt{g}}{e} \right)^{m+1} \left( m + 1 \right) - mb^2 \tag{29}
\]

Hence \( H(x, p) = \frac{1}{2} F^2 \) is given by

\[
H(x, p) = \frac{1}{2} \beta^{*2} \left( \frac{f \pm \beta^* \sqrt{g}}{e} \right)^{m+1} \left( m + 1 \right) \left( \frac{f \pm \beta^* \sqrt{g}}{e} \right)^{m+1} - mb^2 \tag{30}
\]
Putting $\beta^* = b^i p_i$, $\alpha^* = (a^{ij}(x)p_ip_j)^{\frac{1}{2}}$ in (30), we get

$$H(x, p) = \frac{1}{2} \left\{ \frac{(b^i p_i)^2 \left( \frac{f \pm (b^i p_i) \sqrt{g}}{e} \right)^{m+1}}{(m + 1) \left( \frac{f \pm (b^i p_i) \sqrt{g}}{e} - mb^2 \right)^2} \right\}^2$$

(31)

Hence we have the following

**Theorem 3.1.** Let $(M, F)$ be a generalized m-Kropina space of order two and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of $b_i$. then:
1. If $b^2 = 1$, the L-dual of $(M, F)$ is a space on $T^*M$ having the fundamental function:

$$H(x, p) = \frac{1}{2} \left\{ \frac{\beta^{*2} \left( \frac{c \pm (\beta^*) \sqrt{d}}{e} \right)^{m+1}}{(m + 1) \left( \frac{c \pm (\beta^*) \sqrt{d}}{e} - m \right)^2} \right\}^2$$

(32)

where

$$\beta^* = (b^i p_i),$$
$$c = 2m\alpha^* - (m - 1)\beta^*,$$
$$d = 4m\alpha^* + (m - 1)^2 \beta^*,$$
$$e = 2(m + 1)\alpha^*$$

2. If $b^2 \neq 1$, the L-dual of $(M, F)$ is a space on $T^*M$ having the fundamental function:

$$H(x, p) = \frac{1}{2} \left\{ \frac{\beta^{*2} \left( \frac{f \pm \beta^* \sqrt{g}}{e} \right)^{m+1}}{(m + 1) \left( \frac{f \pm \beta^* \sqrt{g}}{e} - mb^2 \right)^2} \right\}^2$$

(33)

where

$$\beta^* = (b^i p_i),$$
$$c = 2mb^2\alpha^* - (m - 1)\beta^*,$$
$$d = 4mb^2\alpha^* + (m - 1)^2 \beta^*,$$
$$e = 2(m + 1)\alpha^*$$
References


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