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The L-Dual of a Generalized m-Kropina Space of Order Two

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Abstract

In ([8]), ([5]) the prolongation to Osc^2M of Riemannian, Finslerian and Lagrangian structures were introduced. The L-dual of a generalized m-Kropina space was introduced in [12]. In this paper we give the L-dual of a generalized m-Kropina space of second order using Lagendre transformation of second order.

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1 Introduction

The L-duality of Finsler and Lagrange spaces was introduced by R. Miron ([7]) and was intensively studied by others, including the first author of this article.

Concrete cases of Hamiltonians obtained by L-duality methods were also constructed. In special the L-duals of some (α, β)-metrics like Randers and Kropina are quite interesting ([2], [3]). The L-dual of another famous $(\alpha, \beta)-$metric, namely the Matsumoto metric was introduced in [9]. In [10], the L-dual of an

(α,β) Finsler space of order two was introduced by I. M. Masca and others. Moreover, very recently, the L-dual of a generalized m-Kropina space was introduced in [12].

A natural questions arises: what is the L-dual of a generalized m-Kropina space of second order. In this paper this is the question we are going to answer.

2 The Legendre transformation

Let us consider a Lagrange space of order two ([5]) denoted by $L^{(2)n}=(M,L(x,y^{(1)},y^{(2)}))$, where $L:(x,y^{(1)},y^{(2)})\in T^2M\longrightarrow L(x,y^{(1)},y^{(2)})\in R$ is the fundamental function and c_{ij} is the fundamental metric tensor given by:

$$c_{ij}(x,y^{(1)},y^{(2)})=\frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}. \tag{1}$$

If M is paracompact manifold, the existence of second order Lagrange spaces, with positively defined fundamental tensor field is always assured ([8]). In this case, there is also a Riemannian metric a on M. Then, the Liouville d-vector

$$z^{(2)i}=y^{(2)i}+\frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k}, \tag{2}$$

is globally defined on $\tilde{E}$, where

$$\tilde{E}=Osc^2M\setminus\{0\}=\left\{(x,y^{(1)},y^{(2)})\in Osc^2M|rank||y^{(1)i}||=1\right\} \tag{3}$$

and it depends only on the metric a. Here γ^i_{jk} are the Christoffel symbols of Riemannian metric a.
The Liouville d-vector $z^{(2)i}$ allows us to construct not only the regular Lagrangian:

$$L(x,y^{(1)},y^{(2)})=\frac{1}{2}\left(a_{ij}(x)z^{(2)i}z^{(2)j}\right)^2, \tag{4}$$

but also some others, for example, putting $\alpha^2=a_{ij}(x)z^{(2)i}z^{(2)j}$ and $\beta=b_i(x)z^{(2)i}$ a differential linear function in $z^{(2)i}$. This is the Prolongation of a Riemannian metric to Osc^2M, introduced by R. Miron in ([7]).

It is known, ([6]), a Finsler space of order two $F^{(2)n}=(M,F(x,y^{(1)},y^{(2)}))$ is a Lagrange space of second order $L^{(2)n}=(M,L(x,y^{(1)},y^{(2)}))$ with

$$L(x,y^{(1)},y^{(2)})=F^2(x,y^{(1)},y^{(2)}), \tag{5}$$

having the fundamental function F positively, 2-homogeneous with respect to $y^{(2)i}$, the fundamental tensor c_{ij} positively defined. In this way, we can define

an (α, β) **Finsler space of order two** as follows:
1. **A Randers space of second order** having the fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \alpha(x, y^{(1)}, y^{(2)}) + \beta(x, y^{(1)}, y^{(2)}), \tag{6}$$

2. **A Kropina space of order two** with fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\beta(x, y^{(1)}, y^{(2)})}, \tag{7}$$

3. **A Matsumoto space of order two** with fundamental function:

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2(x, y^{(1)}, y^{(2)})}{\alpha(x, y^{(1)}, y^{(2)}) - \beta(x, y^{(1)}, y^{(2)})}. \tag{8}$$

4. **A Generalized m-Kropina space of order two** with:

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^{m+1}(x, y^{(1)}, y^{(2)})}{\beta^m(x, y^{(1)}, y^{(2)})} \tag{9}$$

The fundamental function is called, like in classical case, an (α, β)-metric if F is homogeneous of α and β of degree two.

Let us consider a Hamilton space of order two $H^{(2)n} = (M, H(x, y, p))$ with the regular Hamiltonian $H : T^2M \longrightarrow R$, differentiable on $T^{*2}M$ and continuous on the zero section of the projection $\pi^{*2} : T^{*2}M \longrightarrow M$, having the fundamental tensor field:

$$g^{ij}(x, y, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}, \tag{10}$$

with constant signature on the manifold $T^{*2}M$.
Let $C^{(2)n} = (M, K(x, y, p))$ be a Cartan space of order two. From ([8]) it is known that it is a Hamilton space of second order $H^{(2)n}$ for which the fundamental function $H(x, y, p)$ is 2-homogeneous with respect to momenta p_i and

$$H(x, y, p) = K^2(x, y, p). \tag{11}$$

3 The L-dual of a generalized m-Kropina space of order two

In this case, we put: $\alpha^2 = y_i y^j$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b_i p^i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$. We have $F = \frac{\alpha^{m+1}}{\beta^m}$, and

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(2)i}} = F\dot{\partial}_i\left(\frac{\alpha^{m+1}}{\beta^m}\right)$$

which on simplification gives

$$p_i = F\left[\frac{(m+1)F}{\alpha^2}a_{ij}z^{(2)j} - \frac{mF}{\beta}b_i\right]. \tag{12}$$

Contracting in (12) by p^i and b^i respectively, we get

$$\alpha^{*2} = F\left[(m+1)F\left(\frac{\alpha^{2m}}{\beta^{2m}}\right) - m\beta^*\left(\frac{\alpha^{m+1}}{\beta^{m+1}}\right)\right] \tag{13}$$

and

$$\beta^* = F\left[(m+1)\frac{\alpha^{m-1}}{\beta^{m-1}} - mb^2\frac{\alpha^{m+1}}{\beta^{m+1}}\right] \tag{14}$$

A generalized m-Kropina space of order two is a special (α, β)-metric with $\phi = \frac{1}{s^m}$.
Using Shen's [13] notation $s = \frac{\beta}{\alpha}$ above equation becomes :

$$\alpha^{*2} = F\left[\frac{(m+1)F}{s^{2m}} - \frac{m\beta^*}{s^{m+1}}\right] \tag{15}$$

and

$$\beta^* = F\left[\frac{(m+1)}{s^{m-1}} - \frac{mb^2}{s^{m+1}}\right] \tag{16}$$

Putting $s^m = t$, so that $s = t^{\frac{1}{m}}$ in (15) and (16) ,we get

$$\alpha^{*2} = \frac{(m+1)F^2}{t^2} - \frac{mF\beta^*}{t^{\frac{m+1}{m}}} \tag{17}$$

and

$$\beta^* = \frac{(m+1)F}{t^{\frac{m-1}{m}}} - \frac{mFb^2}{t^{\frac{m+1}{m}}} \tag{18}$$

From (18), we get

$$\beta^* t^2 = F\left[(m+1)t^{\frac{m+1}{m}} - mb^2 t^{\frac{m-1}{m}}\right] \tag{19}$$

For $b^2 = 1$, from (19), we get

$$F = \frac{\beta^* t}{(m+1)t^{\frac{1}{m}} - mt^{-\frac{1}{m}}} \tag{20}$$

put the value of F in (17), we get

$$\alpha^{*2}(m+1)^2 t^{\frac{4}{m}} - \left\{2m(m+1)\alpha^{*2} - (m^2-1)\beta^{*2}\right\} t^{\frac{2}{m}} + m^2(\alpha^{*2} - \beta^{*2}) = 0$$

or

$$\alpha^{*2}(m+1)^2 s^4 - \left\{2m(m+1)\alpha^{*2} - (m^2-1)\beta^{*2}\right\} s^2 + m^2(\alpha^{*2} - \beta^{*2}) = 0 \tag{21}$$

solving (21), we get

$$s^2 = \frac{1}{2}\frac{1}{\alpha^{*2}(m+1)}\left\{ \left(2m\alpha^{*2} - (m-1)\beta^{*2}\right) \pm \beta^* \sqrt{4m\alpha^{*2} + (m-1)^2\beta^{*2}} \right\}$$

$$\Rightarrow s = \left[\sqrt{\frac{2m\alpha^{*2} - (m-1)\beta^{*2} \pm \beta^* \sqrt{4m\alpha^{*2} + (m-1)^2\beta^{*2}}}{2(m+1)\alpha^{*2}}} \right]$$

Hence, we get

$$s = \sqrt{\frac{c \pm \beta^* \sqrt{d}}{e}}, \tag{22}$$

where

$$c = 2m\alpha^{*2} - (m-1)\beta^{*2},$$
$$d = 4m\alpha^{*2} + (m-1)^2\beta^{*2},$$
$$e = 2(m+1)\alpha^{*2}$$

Using (22) in (20), we get

$$F = \frac{\beta^* \left(\frac{c \pm \beta^* \sqrt{d}}{e}\right)^{\frac{m+1}{2}}}{(m+1)\left(\frac{c \pm \beta^* \sqrt{d}}{e}\right) - m} \tag{23}$$

Hence $H(x,p) = \frac{1}{2}F^2$ is given by

$$H(x,p) = \frac{1}{2}\frac{\beta^{*2} \left(\frac{c \pm \beta^* \sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c \pm \beta^* \sqrt{d}}{e}\right) - m\right\}^2} \tag{24}$$

Putting $\beta^* = b^i p_i,\ \alpha^* = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$ in (24), we get

$$H(x,p) = \frac{1}{2}\frac{(b^i p_i)^2 \left(\frac{c \pm (b^i p_i) \sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c \pm (b^i p_i) \sqrt{d}}{e}\right) - m\right\}^2} \tag{25}$$

Next, we find $H(x,p)$ for $b^2 \neq 1$, From (19), we have

$$F = \frac{\beta^* t}{(m+1)t^{\frac{1}{m}} - mb^2 t^{-\frac{1}{m}}} \tag{26}$$

Using (26) in (17), we get

$$\alpha^{*2}(m+1)^2 t^{\frac{4}{m}} - \left\{2m(m+1)\alpha^{*2}b^2 - (m^2-1)\beta^{*2}\right\} t^{\frac{2}{m}} + m^2(\alpha^{*2}b^2 - \beta^{*2}) = 0$$

or

$$\alpha^{*2}(m+1)^2 s^4 - \left\{2m(m+1)b^2\alpha^{*2} - (m^2-1)\beta^{*2}\right\} s^2 + m^2(\alpha^{*2}b^2 - \beta^{*2}) = 0 \tag{27}$$

solving (27), we get

$$s^2 = \frac{1}{2}\frac{1}{\alpha^{*2}(m+1)}\left\{2m\alpha^{*2}b^2 - (m-1)\beta^{*2}) \pm \beta^* \sqrt{4mb^2\alpha^{*2} + (m-1)^2\beta^{*2}}\right\}$$

$$\Rightarrow s = \left[\sqrt{\frac{2m\alpha^{*2}b^2 - (m-1)\beta^{*2} \pm \beta^* \sqrt{4m\alpha^{*2}b^2 + (m-1)^2\beta^{*2}}}{2(m+1)\alpha^{*2}}}\right]$$

Hence, we get

$$s = \sqrt{\frac{f \pm \beta^* \sqrt{g}}{e}}, \tag{28}$$

where

$$f = 2mb^2\alpha^{*2} - (m-1)\beta^{*2},$$
$$g = 4mb^2\alpha^{*2} + (m-1)^2\beta^{*2},$$
$$e = 2(m+1)\alpha^{*2}$$

Using (28) in (26), we get

$$F = \frac{\beta^* \left(\frac{f \pm \beta^* \sqrt{g}}{e}\right)^{\frac{m+1}{2}}}{(m+1)\left(\frac{f \pm \beta^* \sqrt{g}}{e}\right) - mb^2} \tag{29}$$

Hence $H(x,p) = \frac{1}{2}F^2$ is given by

$$H(x,p) = \frac{1}{2}\frac{\beta^{*2}\left(\frac{f \pm \beta^* \sqrt{g}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{f \pm \beta^* \sqrt{g}}{e}\right) - mb^2\right\}^2} \tag{30}$$

Putting $\beta^* = b^i p_i,\ \alpha^* = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$ in (30), we get

$$H(x,p) = \frac{1}{2}\frac{(b^i p_i)^2\left(\frac{f \pm (b^i p_i)\sqrt{g}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{f \pm (b^i p_i)\sqrt{g}}{e}\right) - mb^2\right\}^2} \tag{31}$$

Hence we have the following

Theorem 3.1. *Let (M,F) be a generalized m-Kropina space of order two and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of b_i. then:*
1. If $b^2 = 1$, the L-dual of (M,F) is a space on T^*M having the fundamental function:

$$H(x,p) = \frac{1}{2}\frac{\beta^{*2}\left(\frac{c \pm (\beta^*)\sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c \pm (\beta^*)\sqrt{d}}{e}\right) - m\right\}^2} \tag{32}$$

where

$$\begin{aligned} \beta^* &= (b^i p_i),\\ c &= 2m\alpha^{*2} - (m-1)\beta^{*2},\\ d &= 4m\alpha^{*2} + (m-1)^2\beta^{*2},\\ e &= 2(m+1)\alpha^{*2} \end{aligned}$$

2. If $b^2 \neq 1$, the L-dual of (M,F) is a space on T^*M having the fundamental function:

$$H(x,p) = \frac{1}{2}\frac{\beta^{*2}\left(\frac{f \pm \beta^*\sqrt{g}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{f \pm \beta^*\sqrt{g}}{e}\right) - mb^2\right\}^2} \tag{33}$$

where

$$\begin{aligned} \beta^* &= (b^i p_i),\\ c &= 2mb^2\alpha^{*2} - (m-1)\beta^{*2},\\ d &= 4mb^2\alpha^{*2} + (m-1)^2\beta^{*2},\\ e &= 2(m+1)\alpha^{*2} \end{aligned}$$

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