On the Finite Groupoid $Z_n(t, u)$

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Abstract

In this paper we study the existence of commuting regular elements in the groupoid $Z_n(t, u)$. We define the notion left (right) commuting regular elements and study its properties. Also we show that $Z_n(t, u)$ contains commuting regular subsemigroup and give a necessary and sufficient condition for the groupoid $Z_n(t, u)$ to be commuting regular.

Mathematics Subject Classification: 15A27, 20M16, 20L05

Keywords: commuting regular semigroup, semigroup, groupoid

1 Introduction

We use $S$ and $G$ to denote a semigroup and a groupoid, respectively. An element $x$ of a semigroup $S$ is called regular if there exists $y$ in $S$ such that, $x = xyx$ [3]. Two elements $x$ and $y$ of a semigroup $S$ are commuting regular if for some $z \in S$, $xy = yzyx$ [2]. A semigroup $S$ is called commuting regular if and only if for each $x, y \in S$ there exists an element $z$ of $S$ such that $xy = yzyx$ [1]. In [2] we show that the existence of commuting regular elements for the loop ring $Z_t[L_n(m)]$ when $t$ is an even perfect number or $t$ is the form of $2^i p$ or $3^i p$ (where $p$ is an odd prime) or in general when $t = p_1^i p_2$ ( $p_1$ and $p_2$ are distinct odd prime ). Let $Z_n = \{0, 1, 2, \ldots, n-1\}$, $n \geq 3$, for $a, b \in Z_n$ define a binary operation $*$ on $Z_n$ as follows $a * b = ta + ub (\text{ mod } n)$ where $t, u \in Z_n$. In [4, 5] the groupoid ( $Z_n(t, u), *$ ) denote by $Z_n(t, u)$ and study their properties. The groupoid $Z_n(t, u)$ is a semigroup if and only if $t^2 \equiv t(\text{ mod } n)$ and $u^2 \equiv u(\text{ mod } n)$ where $t, u \in Z_n \setminus \{0\}$ and $(t, u) = 1$ [5, Theorem 3.1.1].
2 Commuting Regular Elements

In this section the notion of commuting regular elements of groupoid and commuting regular groupoid are defined. Also, some properties of commuting regular on the groupoid $Z_n(t, u)$ are discussed. The proofs are rather easy and one may use the definitions. However, the important examples are mentioned.

**Definition 2.1** Two elements $x$ and $y$ of a groupoid $G$ are said to be left commuting regular if for some $z \in G$, $xy = ((yx)z)(yx)$. Similarly right commuting regular if for some $z \in G$, $xy = (yx)(z(yx))$ is defined. Finally two elements $x$ and $y$ are commuting regular if they are both left and right commuting regular.

**Definition 2.2** A groupoid $G$ is said to be left commuting regular groupoid if for each $x, y \in G$ there exists $z \in G$ such that $xy = ((yx)z)(yx)$. Similarly, right commuting regular groupoid is defined. A groupoid $G$ is said to be commuting regular groupoid if $G$ is both a left and right commuting regular groupoid.

**Example 2.3** Let $G = \{0, 1, 2\}$ be the groupoid given by the table,

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

then $0$ and $1$ are left commuting regular,

$1=1*0=((0*1)*2)*(0*1)=(2*2)=0*2=1$

but, $(0*1)*(2*(0*1))=2*(2*2)=2*0=2$. Also $0$ and $2$ are right commuting regular,

$1=0*2=(2*0)*(0*(2*0))=2*(0*2)=2*1=1$

but, $((2*0)*0)*(2*0)=(2*0)*2=2*2=0$. Note that $1$ and $1$ are commuting regular.

**Proposition 2.4** Let $(p_1, p_2) = 1$, $p_1$ and $p_2$ are prime integer. Then the groupoid $Z_{p_1 p_2}(1, p_1)$ have commuting regular elements.

**Proof** Suppose that $a = b = kp_2$ where $0 \leq k \leq p_1 p_2$. Therefore

$a^2 = a * a = kp_2 * kp_2 \equiv kp_2 (mod \ p_1 p_2) = a$.

So $a$ is an idempotent element. Then $\{a\}$ is a commuting regular semigroup.

**Corollary 2.5** The groupoid $Z_{p^2}(1, p)$ have commuting regular elements.
**Proof** Suppose that $a = p$ and $b = 0$, then $a * b = (b * a) * 1 * (b * a)$. Also $b * a = (a * b) * p(p - 1) * (a * b)$.

**Proposition 2.6** The groupoid $Z_n(1, p)$ have commuting regular elements where $p|n$.

**Proof** Let $n = tp$. Then $t \in Z_n$ and $t * t = t + tp \equiv t (\text{mod } n)$. Thus $\{t\}$ is a commuting regular semigroup.

**Example 2.7** The groupoid $Z_{10}(1, 5)$ is given by the following table,

<table>
<thead>
<tr>
<th></th>
<th>0 1 2 3 4 5 6 7 8 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 5 0 5 0 5 0 5 0 5</td>
</tr>
<tr>
<td>1</td>
<td>1 6 1 6 1 6 1 6 1 6</td>
</tr>
<tr>
<td>2</td>
<td>2 7 2 7 2 7 2 7 2 7</td>
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<tr>
<td>3</td>
<td>3 8 3 8 3 8 3 8 3 8</td>
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<tr>
<td>4</td>
<td>4 9 4 9 4 9 4 9 4 9</td>
</tr>
<tr>
<td>5</td>
<td>5 0 5 0 5 0 5 0 5 0 5</td>
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<tr>
<td>6</td>
<td>6 1 6 1 6 1 6 1 6 1</td>
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<td>7</td>
<td>7 2 7 2 7 2 7 2 7 2</td>
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<tr>
<td>8</td>
<td>8 3 8 3 8 3 8 3 8 3 8</td>
</tr>
<tr>
<td>9</td>
<td>9 4 9 4 9 4 9 4 9 4 9</td>
</tr>
</tbody>
</table>

Clearly 2, 4, 6 and 8 are idempotent and so $a$ and $b$ are commuting regular elements if $a = b$ and $a \in \{2, 4, 6, 8\}$. Also $G_1 = \{0, 5\}$, $G_2 = \{1, 6\}$, $G_3 = \{2, 7\}$, $G_4 = \{3, 8\}$ and $G_5 = \{4, 9\}$ are commuting regular subsemigroup of $Z_{10}(1, 5)$.

**Proposition 2.8** Let the groupoid $Z_{22p}(1, p)$. Then $\{a\}$ is a commuting regular semigroup where $a \in Z_{22p}$.

**Proposition 2.9** Let the groupoid $Z_{2p}(2p - 2k, 2p - (2k - 1))$ where $0 < k < p$. Then $\{p\}$ is a commuting regular semigroup.

**Proposition 2.10** Let the groupoid $Z_{2p}(2, 2p - 1)$. Then $\{a\}$ is a commuting regular semigroup for all $a \in Z_{2p}$.

**Example 2.11** The groupoid $Z_6(4, 5)$ is given by the following table,

<table>
<thead>
<tr>
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<th>0 1 2 3 4 5</th>
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<tbody>
<tr>
<td>0</td>
<td>0 5 4 3 2 1</td>
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<tr>
<td>1</td>
<td>1 4 3 2 1 0 5</td>
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<tr>
<td>2</td>
<td>2 1 0 5 4 3</td>
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<tr>
<td>3</td>
<td>3 0 5 4 3 2 1</td>
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<tr>
<td>4</td>
<td>4 3 2 1 0 5</td>
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<tr>
<td>5</td>
<td>5 2 1 0 5 4 3</td>
</tr>
</tbody>
</table>
Clearly $G_3 = \{3\}$ is a commuting regular semigroup. Also $G_0 = \{0, 2, 4\}$ and $G_1 = \{1, 3, 5\}$ are left commuting regular sub groupoid of $Z_6(4, 5)$.

**Proposition 2.12** Let the groupoid $Z_{2p}(2p - 2k, 2p + 1 - 2r)$ where $0 < k, r < p$. Then $\{p\}$ is a commuting regular semigroup.

**Proposition 2.13** The gropoid $Z_n(t, u)$ contains a commuting regular subsemigroup, if $t + u \equiv 1 (\mod n)$ where $t, u \in Z_n \setminus \{0\}$ and $t < n$.

**Example 2.14** The groupoid $Z_6(2, 5)$ is given by the following table,

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
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<tr>
<td>1</td>
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<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

Clearly, the subsemigroups $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}$ and $\{5\}$ are commuting regular. Also $G_0 = \{0, 2, 4\}$ and $G_1 = \{1, 3, 5\}$ are commuting regular sub groupoid of $Z_6(2, 5)$.

**Proposition 2.15** The gropoid $Z_n(t, u)$, where $t^2 = t, u^2 = u$ and $t, u \in Z_n \setminus \{0\}$ contain commuting regular elements.

**Proof** Suppose that $a = b = 1$ and $c = n - 2$. Then $a * b = (b * a) * c * (b * a)$.

**Proposition 2.16** Let the groupoid $Z_{p_1 p_2}(p_1(p_2 - 1), p_1 + 1)$. Then $\{a\}$ is a commuting regular semigroup of $Z_{p_1 p_2}(p_1(p_2 - 1), p_1 + 1)$ for all $a \in Z_{p_1 p_2}$.

### 3 Commuting Regular Groupoids

**Theorem 3.1** The groupoid $Z_{2p}(1, p)$ contains a commuting regular sub groupoid.

**Proof** The sub groupoid $G_i = \{i, p + i\}$ given by the following table,

<table>
<thead>
<tr>
<th></th>
<th>i</th>
<th>(p + 1)i</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(p + 1)i$</td>
<td></td>
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</tbody>
</table>

is a commuting regular (where $i = 0, 1, 2, \ldots, p$).
\[ i \ast i = ((p+1)i \ast (p+1)i) \ast (p+1)i \equiv (p+1)i \ast (p+1)i \pmod{2p} \equiv (p+1)i \pmod{2p}. \]

\[ i \ast i = (p+1)i \ast ((p+1)i \ast (p+1)i) \equiv (p+1)i \ast (p+1)i \pmod{2p} \equiv (p+1)i \pmod{2p}. \]

Then \[ G \] clearly is a semigroup of \[ G \].

\[ (p+1)i \ast i = i = i \ast (p+1)i. \]

\[ i = [(i \ast (p+1)i) \ast i] \ast (i \ast (p+1)i) = (i \ast i) \ast i = (p+1)i \ast i \equiv i \pmod{2p}. \]

In the finite groupoid \( Z \mathbb{Z} \) let the groupoid \( \ast \) be regular. Then there exists a commuting regular subgroupoid of \( G \).

\[ a \ast b = ((-2b - a) \ast c) \ast (-2b - a) = (4b + 2a - c) \ast (-2b - a) = -8b - 4a + 2c + 2b + a. \]

Therefore \( (-2a - b) \equiv (-6b - 3a + 2c) \pmod{2p} \). So \( 2c \equiv (5b + a) \pmod{2p} \).

Example 3.2 The groupoid \( Z_0(1, 3) \) is given by the following table.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>0</td>
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<td>2</td>
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</tr>
</tbody>
</table>

Clearly \( G_0 = \{0, 3\} \), \( G_1 = \{1, 4\} \) and \( G_2 = \{2, 5\} \) are commuting regular semigroup of \( G \).

Theorem 3.3 Let the groupoid \( Z_2(2p - 2, 2p - 1) \). Then \( G_0 = \{0, 2, \ldots, 2p - 2\} \) and \( G_1 = \{1, 3, \ldots, 2p - 1\} \) are commuting regular groupoid.

Proof Suppose that \( a, b \in G_1 \). Then \( a \ast b \equiv -2a - b \pmod{2p} \) and \( b \ast a \equiv -2b - a \pmod{2p} \).

\[ a \ast b = ((-2b - a) \ast c) \ast (-2b - a) = (4b + 2a - c) \ast (-2b - a) = -8b - 4a + 2c + b + a. \]

Therefore \( (-2a - b) \equiv (-6b - 3a + 2c) \pmod{2p} \). So \( 2c \equiv (5b + a) \pmod{2p} \).

Thus \( a \ast b = ((b \ast a) \ast c) \ast (b \ast a) \).

\[ a \ast b = (c \ast (-2b - a)) \ast (-2b - a) = (c \ast (-2c + c + 2b + b)) \ast (b \ast a). \]

Thus \( a \ast b = (b \ast a) \ast (a \ast (b \ast a)) \).

Similarly, if \( a \ast b \in G_0 \), then \( a \ast b = (b \ast a) \ast c \ast (b \ast a) \).

Corollary 3.4 Let the groupoid \( Z_2(2p - 2k, 2p + 1 - 2k) \) where \( 0 < k < p \). Then \( G_0 = \{0, 2, \ldots, 2p - 2\} \) and \( G_1 = \{1, 3, \ldots, 2p - 1\} \) are commuting regular groupoid.

Corollary 3.5 Let the groupoid \( Z_2(2p - 2k, 2p - 1 - 2r) \) where \( 0 < k, r < p \). Then \( G_0 = \{0, 2, \ldots, 2p - 2\} \) and \( G_1 = \{1, 3, \ldots, 2p - 1\} \) are commuting regular groupoid.

Theorem 3.6 Let the groupoid \( Z_2(2, 2p - 1) \). Then there exists a commuting regular subgroupoid of \( Z_2(2, 2p - 1) \).
Proof Suppose that $G_0 = \{0, 2, \ldots, 2p - 2\}$ and $G_1 = \{1, 3, \ldots, 2p - 1\}$. Then $G_0$ and $G_1$ are commuting regular subgroupoid. Let $a, b \in G_0 \setminus \{0\}$ and $a > b$. Then $2b - a < a$. We consider two follows case:

1) $2b - a \geq 0$

2) $2b - a < 0$

Case 1. 1) If $2b - a \geq 0$ and $2a - b \leq 2p$, then we show that

$$a \ast b = (b \ast a) \ast c \ast (b \ast a)$$

or $2a - b = (2b - a) \ast c \ast (2b - a) = (4b - 2a - c) \ast (2b - a)$. If $0 \leq 4b - 2a - c \leq 2p$, then $2a - b = (8b - 4a - 2c) - (2b - a)$. Therefore

$$2c \equiv 7b - 5a \pmod{2p}, \text{ if } 0 \leq 6b - 3a - 2c \leq 2p,$$

$$2c \equiv 7b - 5a + 2p \pmod{2p}, \text{ if } 6b - 3a - 2c < 0,$$

$$2c \equiv 7b - 5a - 2p \pmod{2p}, \text{ if } 6b - 3a - 2c > 2p.$$

If $4b - 2a - c > 2p$, then $(2a - b) \equiv (4b - 2a - c - 2p) \ast (2b - a) = (6b - 3a - 2c - 4p).$ Therefore

$$2c \equiv 7b - 5a - 4p \pmod{2p}, \text{ if } 0 \leq 6b - 3a - 2c - 4p \leq 2p,$$

$$2c \equiv 7b - 5a - 2p \pmod{2p}, \text{ if } 6b - 3a - 2c - 4p < 0,$$

$$2c \equiv 7b - 5a - 6p \pmod{2p}, \text{ if } 6b - 3a - 2c - 4p > 2p.$$

If $4b - 2a - c < 0$, then $(2a - b) \equiv (4b - 2a - c + 2p) \ast (2b - a) = (6b - 3a - 2c + 4p).$ Therefore

$$2c \equiv 7b - 5a + 4p \pmod{2p}, \text{ if } 0 \leq 6b - 3a - 2c + 4p \leq 2p,$$

$$2c \equiv 7b - 5a + 6p \pmod{2p}, \text{ if } 6b - 3a - 2c + 4p < 0,$$

$$2c \equiv 7b - 5a + 2p \pmod{2p}, \text{ if } 6b - 3a - 2c + 4p > 2p.$$

Case 1. 2) If $2b - a \geq 0$ and $2a - b > 2p$, then we show that

$$a \ast b = (b \ast a) \ast c \ast (b \ast a)$$

or $a \ast b = 2a - b - 2p = (2b - a) \ast c \ast (2b - a)$. Therefore $2a - b - 2p = (4b - 2a - c) \ast (2b - a)$. If $0 \leq 4b - 2a - c \leq 2p$, then $2c \equiv 7b - 5a + 2p \pmod{2p}$. If $4b - 2a - c > 2p$, then

$$2c \equiv 7b - 5a - 2p \pmod{2p}, \text{ if } 0 \leq 6b - 3a - 2c - 4p \leq 2p.$$
Since $2c \equiv 7b - 5a (\text{mod } 2p)$, if $6b - 3a - 2c - 4p < 0$,
$2c \equiv 7b - 5a - 4p (\text{mod } 2p)$, if $6b - 3a - 2c - 4p > 2p$.

Case 2. 1) If $2b - a < 0$ and $2a - b \leq 2p$, then we show that
$$a \ast b = (b \ast a) \ast c \ast (b \ast a).$$

Since $2b - a \equiv 2b - a + 2p (\text{mod } 2p)$, $2b - a \equiv (2b - a + 2p) \ast c \ast (2b - a + 2p) (\text{mod } 2p)$. If $0 \leq 4p + 4b - 2a - c \leq 2p$, $2b - a \equiv 6p + 6b - 3a - 2a (\text{mod } 2p)$ and so
$$2c \equiv 7b - 5a + 6p (\text{mod } 2p), \text{ if } 0 \leq 6b - 3a - 2c + 6p \leq 2p,$$
$$2c \equiv 7b - 5a + 8p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 6p < 0,$$
$$2c \equiv 7b - 5a + 4p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 6p > 2p.$$

If $4p + 4b - 2a - c > 2p$, then $2b - a \equiv 2p + 6b - 3a - 2c (\text{mod } 2p)$. Therefore
$$2c \equiv 7b - 5a + 2p (\text{mod } 2p), \text{ if } 0 \leq 6b - 3a - 2c + 2p \leq 2p,$$
$$2c \equiv 7b - 5a + 4p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 2p < 0,$$
$$2c \equiv 7b - 5a (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 2p > 2p.$$

If $4p + 4b - 2a - c < 0$, then $2b - a \equiv 10p + 6b - 3a - 2c (\text{mod } 2p)$. Therefore
$$2c \equiv 7b - 5a + 10p (\text{mod } 2p), \text{ if } 0 \leq 10b - 3a - 2c + 2p \leq 2p,$$
$$2c \equiv 7b - 5a + 12p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 10p < 0,$$
$$2c \equiv 7b - 5a + 8p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 10p > 2p.$$

Case 2. 2) If $2b - a < 0$ and $2a - b > 2p$, then we show that
$$a \ast b = (b \ast a) \ast c \ast (b \ast a).$$

If $4p + 4b - 2a - c \leq 2p$, then $2a - b - 2p \equiv 6p + 6b - 3a - 2c (\text{mod } 2p)$ and so
$$2c \equiv 7b - 5a + 8p (\text{mod } 2p), \text{ if } 0 \leq 6b - 3a - 2c + 6p \leq 2p,$$
$$2c \equiv 7b - 5a + 10p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 6p < 0,$$
$$2c \equiv 7b - 5a + 6p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 6p > 2p.$$

If $4p + 4b - 2a - c > 2p$, then $2a - b - 2p \equiv 4p + 6b - 3a - 2c (\text{mod } 2p)$ and so
$$2c \equiv 7b - 5a + 6p (\text{mod } 2p), \text{ if } 0 \leq 6b - 3a - 2c + 4p \leq 2p,$$
$$2c \equiv 7b - 5a + 8p (\text{mod } 2p), \text{ if } 6b - 3a - 2c + 4p < 0,$$
2c ≡ 7b − 5a + 4p (mod 2p), if 6b − 3a − 2c + 4p > 2p.

If 4p + 4b − 2a − c < 0, then 2a − b − 2p ≡ 12p + 6b − 3a − 2c (mod 2p) and so

2c ≡ 7b − 5a + 14p (mod 2p), if 0 ≤ 6b − 3a − 2c + 12p ≤ 2p,

2c ≡ 7b − 5a + 16p (mod 2p), if 6b − 3a − 2c + 12 < 0,

2c ≡ 7b − 5a + 12p (mod 2p), if 6b − 3a − 2c + 12p > 2p.

Similarly, if a < b, then we have a * b = (b * a) * c * (b * a).

Finally, if a = b, then a * a = (a * a) * c * (a * a) where a = c.

Example 3.7 The groupoid $\mathbb{Z}_{10}(2, 9)$ is given by the following table,

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
1 & 2 & 1 & 0 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\
2 & 4 & 3 & 2 & 1 & 0 & 9 & 8 & 7 & 6 & 5 \\
3 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 9 & 8 & 7 \\
4 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 9 \\
5 & 0 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
6 & 2 & 1 & 0 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\
7 & 4 & 3 & 2 & 1 & 0 & 9 & 8 & 7 & 6 & 5 \\
8 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 9 & 8 & 7 \\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 9 \\
\end{array}
\]

then $G_0 = \{0, 2, 4, 6, 8\}$ and $G_1 = \{1, 3, 5, 7, 9\}$ are commuting regular sub groupoid.

Theorem 3.8 The groupoid $\mathbb{Z}_{n}(t, t)$ is a commuting regular groupoid where $t^2 \equiv t \ (mod \ n)$ and $t < n$.

Proof For $a, b \in \mathbb{Z}_n$, $a * b = b * a = ta + tb = t(a + b)$. So $\mathbb{Z}_n(t, t)$ is a commutative. Now, $a * b = (b * a) * c * (b * a) = t(a + b) * c * t(a + b) \equiv t(a + b + c) * t(a + b)(mod \ n) \equiv t(2a + 2b + c)(mod \ n)$. By $c = n - (a + b)$, $a * b = (b * a) * c * (b * a)$.

Example 3.9 The groupoid $\mathbb{Z}_6(3, 3)$ is given by the following table,

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 3 & 0 & 3 & 0 \\
1 & 3 & 0 & 3 & 0 & 3 \\
2 & 0 & 3 & 0 & 3 & 0 \\
3 & 3 & 0 & 3 & 0 & 3 \\
4 & 0 & 3 & 0 & 3 & 0 \\
5 & 3 & 0 & 3 & 0 & 3 \\
\end{array}
\]
is a commuting regular.

**Theorem 3.10** The groupoid $Z_n(t, t)$ is a commuting regular semigroup where $t^2 \equiv t \pmod{n}$ and $t + t \equiv 0 \pmod{n}$.

**Proof** For $a, b \in Z_n$, $a \ast b = ta + tb = b \ast a$. Let $c = a + b$. Then $a \ast b = (b \ast a) \ast c = (b \ast a) = t(a + b) \ast c = t(a + b) = t((t + t)b + t)c \equiv [(t + t)a + (t + t)b + tc] \pmod{n}$.

**Theorem 3.11** The groupoid $Z_n(t, t)$ is a commuting regular semigroup where $t^2 \equiv t \pmod{n}$ and $t + t \equiv 1 \pmod{n}$.

**Proof** For $a, b \in Z_n$, $a \ast b = ta + tb = b \ast a$. If $c = 0$, then $a \ast b = (b \ast a) \ast c = (b \ast a) = t(a + b) \ast c = t(a + b) = t((t + t)b + 0)c \equiv [(t + t)a + (t + t)b] \pmod{n}$.

**Proposition 3.12** Let $n = kt$. Then $G = \{0, k\}$ is a commuting regular sub semigroup of the groupoid $Z_n(t, 0)$.

**Proof** The semigroup $G$ given by following table,

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>k</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

hence $G$ is a commuting regular.

**Corollary 3.13** Let $n = kt$. Then $G = \{0, k\}$ is a commuting regular sub semigroup of the groupoid $Z_n(0, t)$.

**Theorem 3.14** Let $t^2 \equiv t \pmod{n}$, $u^2 \equiv u \pmod{n}$ and $(t, u) = 1$ for $t, u \in Z_n \setminus \{0\}$. Then the groupoid $Z_n(t, u)$ is a Van Neumman semigroup if $t + u \equiv 1 \pmod{n}$ and $\{a\}$ is commuting regular for $a \in Z_n(t, u)$.

**Proof** For all $a \in Z_n$, $a^2 = a \ast a = ta + ua = (t + u)a \equiv a \pmod{n}$. So $a^2 = a$ and $a^2 = a^2 \ast a \ast a^2$. Therefore $\{a\}$ is a commuting regular. Since $t + u \equiv 1 \pmod{n}$, $tu \equiv 0 \pmod{n}$ and so $(a \ast c) \ast a = (ta + uc) \ast a = t^2a + tuc + ua \equiv (ta + ua) \pmod{n} \equiv a \pmod{n}$. Thus $Z_n(t, u)$ is a Van Neumman semigroup.

**Theorem 3.15** Let $t^2 \equiv t \pmod{n}$, $u^2 \equiv u \pmod{n}$, $(t, u) = 1$ and $t + u \equiv 1 \pmod{n}$ for $t, u \in Z_n \setminus \{0\}$. Then the groupoid $Z_n(t, u)$ is a commutative semigroup if and only if $Z_n(t, u)$ is a commuting regular semigroup.
Proof Suppose that $Z_n(t, u)$ be a commuting regular semigroup. Then for $a, b \in Z_n$ there exists $c \in Z_n$ such that $a * b = (b * a) * c * (b * a)$.

So $a * b = (tb + ua) * c * (tb + ua) = (t^2b + tua + uc) * (tb + ua) \equiv (tb + uc) * (tb + ua) \mod n \equiv t^2b + tua + uc * (tb + ua) \equiv tb + ua \mod n = b * a$. On the others hand, if $a * b = b * a$, then by the Proposition 3.10 there exists $c \in Z_n$ such that $a * b = (b * a) * c * (b * a)$.

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References


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