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Hyperidentities in Graph Variety Generated by ((XX)(Y((ZX)Z)))Z Graph

P. Jampachon and T. Poomsa-ard

Department of Mathematics, Faculty of Science
Khonkaen University, Khonkaen 41000, Thailand
prajam@kku.ac.th, tiang@kku.ac.th

Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. The set of all term equations $s \approx t$ which the graph G satisfies denoted by $Id(\{G\}$. The class of all graph algebras satisfy for all term equations in $Id(\{G\}$ is called the graph variety generated by $\{G\}$ denoted by $\mathcal{V}_g(\{G\})$. A term equation $s \approx t$ is called an identity $\mathcal{V}_g(\{G\})$ if $\underline{A(G)}$ satisfies $s \approx t$ for all $G \in \mathcal{V}_g(\{G\})$. An identity $s \approx t$ of a $\mathcal{V}_g(\{G\})$ is called a hyperidentity of a graph algebra $\underline{A(G')}$, $G' \in \mathcal{V}_g(\{G\})$ whenever the operation symbols occuring in s and t are replaced by any term operations of $\underline{A(G)}$ of the appropriate arity, the resulting identities hold in $\underline{A(G)}$. An identity $s \approx t$ of a $\mathcal{V}_g(\{G\})$ is called a hyperidentity of $\mathcal{V}_g(\{G\})$ if it is a hyperidentity of $\underline{A(G')}$ for all $G' \in \mathcal{V}_g(\{G\})$.

In this paper we characterize all identities and all hyperidentities in $\mathcal{V}_g(\{G\})$ where G is the $((xx)(y((zx)z)))z$ graph. Some applications and examples are considered.

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1 Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in [1].

We fix a type $\tau = (n_i)_{i\in I},\ n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i\in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then, a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). We denote the extension of the hypersubstitution σ by a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras were invented by Shallon in [12], to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In a study by Pöschel and Wessel [11], graph varieties were investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [10], these investigations were extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

Let $s \approx t$ be a term equation. In [3], [7], [8], T. Poomsa-ard et.al. characterized the graph variety $\mathcal{V} = Mod_g(\{s \approx t\})$ in various kind of terms s and t. Further they characterized identities and hyperidentities in these graph varieties, too. But these results are not convenient for apply to the real-world situation. Because at first we will check that what kind of terms s and t which the graph variety $\mathcal{V} = Mod_g(\{s \approx t\})$ contains the graph algebra of the diagram of that real-world situation. It is not easy to do this. So we will characterize the graph variety generate by the graph G of the diagram directly. Then characterize identities and hyperidentities of this graph variety.

In this paper we characterized all identities and all hyperidentities in $\mathcal{V}_g(\{G\})$ where G is the $((xx)(y((zx)z)))z$ graph.

2 Terms, identities and graph varieties

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞ too).

Definition 2.1. A term over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...$, and ∞ are terms;
(ii) if t_1 and t_2 are terms, then t_1t_2 is a term.

$W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps. Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$, the rightmost variable of a term t is denoted by $R(t)$. A term in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$, one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies a term equation* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to \overline{V \cup \{\infty\}}$), and in this case, we write

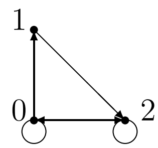
$G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of term equations (i.e., $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,
$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,
$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,
$Id\mathcal{G} = \{s \approx t \mid s, t \in W_\tau(X),\ \mathcal{G} \models s \approx t\}$,
$Mod_g\Sigma = \{G \mid G$ is a graph and $G \models \Sigma\}$,
$\mathcal{V}_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of term equations such that $\mathcal{G} = \mathrm{Mod}_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

3 Identities in graph variety generated by $((xx)(y((zx)z)))z$ graph

The following is the diagram of $((xx)(y((zx)z)))z$ graph:



G

Let $\mathcal{K}' = \mathcal{V}_g(\{G\})$. We want to characterized all identities in $\mathcal{K}'$. Before to do this we need some results for reference as the following:

Graph identities were characterized in [4] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras if and only if either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further they proved.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Proposition 3.3. *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

Now we characterize all identities in $\mathcal{K}'$. Clearly, if $s \approx t$ is a trivial equation (i.e. s and t are both trivial terms or $G(s) = G(t)$, $L(s) = L(t)$), then $s \approx t$ is an identity in $\mathcal{K}'$. Now we consider the case $s \approx t$ is a non-trivial equation. Then all identities in $\mathcal{K}'$ are characterized by the following theorem:

Theorem 3.1. *Let $s \approx t$ be non-trivial equation. Then $G \models s \approx t$ if and only if the following conditions are satisfied:*

(i) $L(s) = L(t)$ and $V(s) = V(t)$,

(ii) there exists $y \in V(s)$ such that $(y, L(s)) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(z, L(t)) \in E(t)$,

(iii) for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(x, z) \in E(t)$, in particular $(x, y), (y, x) \in E(s)$ if and only if $(x, z), (z, x) \in E(t)$,

(iv) for any $x, y \in V(s)$ with $x \neq y$,

(a) if there exists no $z \in V(s)$ such that $(y, z), (z, y) \in E(s)$, then $(x, y) \in E(s)$ if and only if $(x, y) \in E(t)$,

(b) if there exists $z \in V(s)$ such that $(y, z), (z, y) \in E(t)$, then $(x, y) \in E(s)$ or there exists $w \in V(s)$ such that $(x, w), (w, x) \in E(s)$ if and only if $(x, y) \in E(t)$ or there exists $w' \in V(s)$ such that $(x, w'), (w', x) \in E(t)$.

Proof. To prove (i). Suppose that $V(s) \neq V(t)$, we can let $x \in V(s)$ but $x \notin V(t)$. Let $h : V(s) \cup V(t) \longrightarrow V \cup \{\infty\}$ such that $h(x) = \infty$ and $h(y) = 0$ for all other $y \in V(s) \cup V(t)$. We have $h(s) = \infty$ and $h(t) = 0$. Hence $G \not\models s \approx t$. We get $V(s) = V(t)$. Suppose that $L(s) \neq L(t)$. Let $h : V(s) \longrightarrow V \cup \{\infty\}$ such that $h(L(s)) = 0$ and $h(y) = 2$ for all other $y \in V(s)$. We have $h(s) = h(L(s)) = 0$ and $h(t) = h(L(t)) = 2$. Hence $G \not\models s \approx t$. We get $L(s) = L(t)$.

To prove (ii). Suppose that there exists $y \in V(s)$ such that $(y, L(s)) \in E(s)$ but there exists no $z \in V(s)$ such that $(z, L(t)) \in E(t)$. Let $h : V(s) \longrightarrow V$ such that $h(L(s)) = 1$, $h(y) = 2$ for all other $y \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t)) = 1$. Hence $G \not\models s \approx t$.

To prove (iii). Suppose that there is some $x \in V(s)$ which there exists $y \in V(s)$ such that $(x, y) \in E(s)$ but there exists no $z \in V(s)$ such that $(x, z) \in E(t)$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 0$ for all other $y \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t)) = 0$. Hence $G \not\models s \approx t$. Suppose that $(x, y), (y, x) \in E(s)$ but there exists no $z \in V(s)$ such that $(x, z), (z, x) \in E(t)$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(u) = 0$ for all $u \in V(s)$ with $(u, x) \in E(s)$ and $h(v) = 2$ for all other $v \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $G \not\models s \approx t$.

To prove $(iv)(a)$. Suppose that there are $x, y \in V(s)$ with $x \neq y$, there exists no $z \in V(s)$ such that $(y, z), (z, y) \in E(s)$ and $(x, y) \in E(s)$ but $(x, y) \notin E(t)$. By (iii) we have there exists no $w \in V(s)$ such that $(y, w), (w, y) \in E(t)$. Let $h : V(s) \longrightarrow V$ such that $h(y) = 1$, $h(u) = 0$ for all $u \in V(s)$ with $(u, y) \in E(t)$ and $h(v) = 2$ for all other $v \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $G \nvDash s \approx t$.

To prove $(iv)(b)$. Suppose that there are $x, y \in V(s)$ with $x \neq y$, there exists $z \in V(s)$ such that $(y, z), (z, y) \in E(s)$ and $(x, y) \in E(s)$ or there exists $w \in V(s)$ such that $(x, w), (w, x) \in E(s)$. If there exists $w \in V(s)$ such that $(x, w), (w, x) \in E(s)$, then by (iii) we have there exists $w' \in V(s)$ such that $(x, w'), (w', x) \in E(t)$. Suppose that $(x, y) \in E(s)$ but $(x, y) \notin E(t)$ and there exists no $w \in V(s)$ such that $(x, w), (w, x) \in E(t)$. By (iii) we have there exists $z' \in V(s)$ such that $(y, z'), (z', y) \in E(t)$ and there exists no $w' \in V(s)$ such that $(x, w'), (w', x) \in E(s)$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 0$, $h(u) = 0$ for all $u \in V(s)$ with $(u, x) \in E(t)$ and $h(v) = 2$ for all other $v \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $G \nvDash s \approx t$. In the similar way, we can prove the converse.

Conversely, suppose that $s \approx t$ is a non-trivial equation satisfying (i), (ii), (iii) and (iv). Let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (iii) we have there exists $z \in V(s)$ such that $(x, z), (z, x) \in E(s)$. Since h is a homomorphism from $G(s)$ into G, we get $h(x) = 0$ or $h(x) = 2$. Hence $(h(x), h(x)) \in E$. If $x \neq y$ and there exists no $z \in V(s)$ such that $(y, z), (z, y) \in E(t)$, then by $(iv)(a)$ we have $(x, y) \in E(s)$. Hence $(h(x), h(y)) \in E$. If $x \neq y$ and there exists $z \in V(s)$ such that $(y, z), (z, y) \in E(t)$, then by $(iv)(b)$ we have $(x, y) \in E(s)$ or there exists $w \in V(s)$ such that $(x, w), (w, x) \in E(s)$. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. If there exists $w \in V(s)$ such that $(x, w), (w, x) \in E(s)$, then $h(x) = 0$ or $h(x) = 2$ and $h(y) = 0$ or $h(y) = 2$. We get $(h(x), h(y)) \in E$. Therefore h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.3 we get $G \vDash s \approx t$. □

4 Hyperidentities in graph variety generated by $((xx)(y((zx)z)))z$ graph

Let $\mathcal{K}$ be any graph variety. Now, we want to formulate precise the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \rightarrow W_\tau(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* if and only if for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{K}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$ we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions. In [5], the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_\mathcal{K} \sigma_2$.

The following lemma was proven in [6].

Lemma 4.1. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ and $\sigma_1 \sim_\mathcal{K} \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.*

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_\mathcal{K}$.

In [9], R. Pöschel showed that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

The following definition was given in [2].

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_\mathcal{K} \sigma_{NF(t)}$.

In [2], all rooted graphs with at most two vertices were considered. Then, we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given as the following table:

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

By Theorem 3.1 we have the following relations:

(*i*) $\sigma_{10} \sim_{\mathcal{K}'} \sigma_{12} \sim_{\mathcal{K}'} \sigma_{14} \sim_{\mathcal{K}'} \sigma_{16} \sim_{\mathcal{K}'} \sigma_{18}$,

(*ii*) $\sigma_{11} \sim_{\mathcal{K}'} \sigma_{13} \sim_{\mathcal{K}'} \sigma_{15} \sim_{\mathcal{K}'} \sigma_{17} \sim_{\mathcal{K}'} \sigma_{19}$.

Let $M_{\mathcal{K}'}$ be the set of all normal form graph hypersubstitutions in $\mathcal{K}'$. Then we get,

$$M_{\mathcal{K}'} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}\}.$$

We defined the product of two normal form graph hypersubstitutions in $M_{\mathcal{K}'}$ as follows.

Definition 4.5. The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ *of two normal form graph hypersubstitutions* is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The following table gives the multiplication of elements in $M_{\mathcal{K}'}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_2	σ_2	σ_1	σ_2	σ_2
σ_3	σ_3	σ_1	σ_2	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4	σ_3	σ_4
σ_4	σ_4	σ_1	σ_2	σ_3	σ_4	σ_3	σ_4	σ_4	σ_4	σ_3	σ_4	σ_4
σ_5	σ_5	σ_1	σ_2	σ_3	σ_4	σ_0	σ_9	σ_{11}	σ_7	σ_6	σ_{11}	σ_{11}
σ_6	σ_6	σ_1	σ_2	σ_3	σ_4	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}
σ_7	σ_7	σ_1	σ_2	σ_3	σ_4	σ_6	σ_{11}	σ_{11}	σ_7	σ_6	σ_{11}	σ_{11}
σ_8	σ_8	σ_1	σ_2	σ_3	σ_4	σ_9	σ_{10}	σ_{11}	σ_8	σ_9	σ_{10}	σ_{11}
σ_9	σ_9	σ_1	σ_2	σ_3	σ_4	σ_8	σ_9	σ_{11}	σ_{11}	σ_{10}	σ_{11}	σ_{11}
σ_{10}	σ_{10}	σ_1	σ_2	σ_3	σ_4	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}
σ_{11}	σ_{11}	σ_1	σ_2	σ_3	σ_4	σ_{10}	σ_{11}	σ_{11}	σ_{11}	σ_{10}	σ_{11}	σ_{11}

The concept of a proper hypersubstitution of a class of algebras was introduced in [6].

Definition 4.6. A hypersubstitution σ is called *proper with respect to a class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

A graph hypersubstitution with the property $\sigma(f)$ contains both variables x_1 and x_2 is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

The following lemma was proved in [2].

Lemma 4.2. *For each non-trivial term* $s, (s \neq x \in X)$ *and for all* $u, v \in X$, *we have*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(u,u) | (u,v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u) | (u,v) \in E(s)\}.$$

In the similar way we can prove that,

$$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u,u),(v,v)|(u,v) \in E(s)\}.$$

We want to find all proper graph hypersubstitutions with respect to $\mathcal{K}'$. Then we obtain:

Theorem 4.1. $\{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$ *is the set of all proper graph hypersubstitutions with respect to* $\mathcal{K}'$.

Proof. If $s \approx t \in Id\mathcal{K}'$ and s, t are trivial terms, then $\hat{\sigma}_6[s]$, $\hat{\sigma}_8[s]$, $\hat{\sigma}_{10}[s]$, $\hat{\sigma}_6[t]$, $\hat{\sigma}_8[t]$ and $\hat{\sigma}_{10}[t]$ are also trivial terms and thus $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}'$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}'$ and $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{K}'$. In the same manner, we see that $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}'$ and $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{K}'$, if $G(s) = G(t)$ and $L(s) = L(t)$.

Now, assume that $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}'$. Then $(i) - (iv)$ of Theorem 3.1 hold.

For σ_6, σ_8 and σ_{10} we obtain:

$$L(\hat{\sigma}_6[s]) = L(s) = L(t) = L(\hat{\sigma}_6[t],$$

$$L(\hat{\sigma}_8[s]) = L(s) = L(t) = L(\hat{\sigma}_8[t],$$

$$L(\hat{\sigma}_{10}[s]) = L(s) = L(t) = L(\hat{\sigma}_{10}[t].$$

Since σ_6, σ_8 and σ_{10} are regular, we have:

$$V(\hat{\sigma}_6[s]) = V(s) = V(t) = V(\hat{\sigma}_6[t],$$

$$V(\hat{\sigma}_8[s]) = V(s) = V(t) = V(\hat{\sigma}_8[t],$$

$$V(\hat{\sigma}_{10}[s]) = V(s) = V(t) = V(\hat{\sigma}_{10}[t].$$

For σ_6, since $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}'$. Then by Theorem 3.1$(iii)$ and Lemma 4.2, we have for any $x \in V(s)$, $(x,x) \in E(\hat{\sigma}_6[s])$ if and only if $(x,x) \in E(\hat{\sigma}_6[t])$ or there exists no $z \in V(s)$ such that $(x,z) \in E(\hat{\sigma}_6[s])$ if and only if there exists no $w \in V(s)$ such that $(x,w) \in E(\hat{\sigma}_6[t])$. For any $y \in V(s)$ with there exists no $z \in V(s)$ such that $(y,z) \in E(\hat{\sigma}_6[s])$, by Theorem 3.1$(iv)(a)$ and Lemma 4.2, we have $(x,y) \in E(\hat{\sigma}_6[s])$ if and only if $(x,y) \in E(\hat{\sigma}_6[t])$. Then by Theorem 3.1, we get $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}'$.

For σ_8, since $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}'$. Then by Theorem 3.1$(ii)$ and Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_8[s])$ and $(x,x) \in E(\hat{\sigma}_8[t])$ for all $x \in V(s)$, $x \neq L(s)$. For $L(s)$ by Theorem 3.1(ii), we have $(L(s),L(s)) \in E(\hat{\sigma}_8[s])$ if and only if $(L(t),L(t)) \in E(\hat{\sigma}_6[t])$, Then by Theorem 3.1, we get $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}'$.

For σ_{10}, since $s \approx t$ is a non-trivial equation and $s \approx t \in Id\mathcal{K}'$, by Lemma 4.2, we have $(x,x) \in E(\hat{\sigma}_{10}[s])$ and $(x,x) \in E(\hat{\sigma}_{10}[t])$ for all $x \in V(s)$. We get the conditions of Theorem 3.1 hold. Then by Theorem 3.1, we have $\hat{\sigma}_{10}[s] \approx \hat{\sigma}_{10}[t] \in Id\mathcal{K}'$.

For each $\sigma \notin \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$ we give an identity $s \approx t \in Id\mathcal{K}'$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}'$. Clearly, if s and t are trivial terms with different leftmost and different rightmost, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \notin Id\mathcal{K}'$, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \notin Id\mathcal{K}$, $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \notin Id\mathcal{K}'$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \notin Id\mathcal{K}'$.

Now, let $s = (x_1x_1)(x_2x_2)$ and $t = x_1(x_2x_1)$. By Theorem 3.1 we get $s \approx t \in Id\mathcal{K}'$. If $\sigma \in \{\sigma_5, \sigma_7, \sigma_9, \sigma_{11}\}$, then $L(\sigma(f)) = x_2$. We see that $L(\hat{\sigma}[s]) = x_2$ and $L(\hat{\sigma}[t]) = x_1$. Thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}'$. □

Now, we apply our results to characterize all hyperidentities in $\mathcal{K}'$. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{K}'$ if and only if $L(s) = L(t)$, $R(s) = R(t)$ and $x \approx x$, $x \in X$ is a hyperidentity in $\mathcal{K}'$, too. So, we consider the case that s and t are non-trivial terms and different from variables.

Theorem 4.2. *An identity $s \approx t$ in $\mathcal{K}'$, where s and t are non-trivial terms and different from variables is a hyperidentity in $\mathcal{K}'$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is also an identity in $\mathcal{K}'$.*

Proof. If $s \approx t$ is a hyperidentity in $\mathcal{K}'$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}'$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}'$ and that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}'$, too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{K}'}$.

If σ is a proper, then we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}'$. By assumption, $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}'$.

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(\hat{\sigma}_5[s]) = L(\hat{\sigma}_5[t]) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(\hat{\sigma}_5[s])L(\hat{\sigma}_5[s]) = L(\hat{\sigma}_5[t])L(\hat{\sigma}_5[t]) = \hat{\sigma}_4[t]$.

Since $\sigma_6 \circ_N \sigma_5 = \sigma_7$, we have $\hat{\sigma_6}[\hat{\sigma}_5[s]] = \hat{\sigma_7}[s]$ and $\hat{\sigma_6}[\hat{\sigma}_5[t]] = \hat{\sigma_7}[t]$. Because of $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}'$ and σ_6 is a proper, we have $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}'$. In the same way since $\sigma_8 \circ_N \sigma_5 = \sigma_9$ and $\sigma_{10} \circ_N \sigma_5 = \sigma_{11}$, we have $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$ are identities in $\mathcal{K}'$. □

5 Examples and Applications

In this section we give some examples of term equation which are identities in $\mathcal{K}'$ and some examples of term equation which are hyperidentities in $\mathcal{K}'$. After that we give some applications of identities and hyperidentities to some diagrams.

By Theorem 3.1, we have the examples of $s \approx t \in Id\mathcal{K}'$ as the following examples:

Example 5.1. Examples of $s \approx t \in Id\mathcal{K}'$ are given as the following:

(a) Terms with one variable: $(xx) \approx (xx)x$.

(b) Terms with two variables: $(xx)(yy) \approx x(yx)$, $(xx)(yy) \approx (xx)(yx)$, $x((yx)y) \approx (xx)((yx)y)$, $(yy)(xx) \approx y((xx)y)$, $(yy)(xy) \approx (yy)((xx)y)$.

(c) Terms with three variables: $((xx)(yy))(zz) \approx (xx)((yy)((zz)(xz)))$, $((xx)(y(zz)))z \approx ((xx)(y((zx)z)))z$.

By Theorem 4.2, we have the examples of $s \approx t \in Id\mathcal{K}'$ which are hyperidentities in $\mathcal{K}'$ as the following examples:

Example 5.2. Examples of $s \approx t \in Id\mathcal{K}'$ with $s \approx t$ are hyperidentities in $\mathcal{K}'$ are given as the following:

(a) Terms with one variable: $(xx) \approx (xx)x$. Since $\hat{\sigma}_5[(xx)] = (xx)$ and $\hat{\sigma}_5[(xx)x] = x(xx)$, we see that $(xx) \approx x(xx) \in Id\mathcal{K}'$.

(b) Terms with two variables: $x((yx)y) \approx (xx)((yx)y)$. Since $\hat{\sigma}_5[x((yx)y)] = (y(xy))x$ and $\hat{\sigma}_5[(xx)((yx)y)] = (y(xy))(xx)$, we see that $(y(xy))x \approx (y(xy))(xx) \in Id\mathcal{K}'$.

(c) Terms with three variables: $((xx)(yy))(zz) \approx (xx)((yy)((zz)(xz)))$. Since $\hat{\sigma}_5[((xx)(yy))(zz)] = (zz)((yy)(xx))$ and $\hat{\sigma}_5[(xx)((yy)((zz)(xz)))] = (((zx)(zz))(yy))(xx)$, we see that $(zz)((yy)(xx)) \approx (((zx)(zz))(yy))(xx) \in Id\mathcal{K}'$.

By definition of proper graph hypersubstitution, we see that if $s \approx t \in Id\mathcal{K}'$, then we can find the others identities in $\mathcal{K}'$ by apply some proper graph hypersubstitutions to $s \approx t$ as the following examples:

Example 5.3. In Example 5.1 we know that $(xx)(yy) \approx x(yx) \in Id\mathcal{K}'$, we can get other identities in $\mathcal{K}'$ as the followings:

(a) Apply σ_6 to $(xx)(yy) \approx x(yx)$, we get

$\hat{\sigma}_6[(xx)(yy)] = (((xx)x)((xx)x))((yy)y)$ and $\hat{\sigma}_6[x(yx)] = (xx)((yy)x)$. Hence $(((xx)x)((xx)x))((yy)y) \approx (xx)((yy)x) \in Id\mathcal{K}'$.

(b) Apply σ_8 to $(xx)(yy) \approx x(yx)$, we get

$\hat{\sigma}_8[(xx)(yy)] = (x(xx))((y(yy))(y(yy)))$ and $\hat{\sigma}_8[x(yx)] = x((y(xx))(y(xx)))$. Hence $(x(xx))((y(yy))(y(yy))) \approx x((y(xx))(y(xx))) \in Id\mathcal{K}'$.

(c) Apply σ_{10} to $(xx)(yy) \approx x(yx)$, we get

$\hat{\sigma}_{10}[(xx)(yy)] = (((xx)(xx))((xx)(xx)))(((yy)(yy))((yy)(yy)))$ and

$\hat{\sigma}_{10}[x(yx)] = (xx)(((yy)(xx))((yy)(xx)))$.

Hence $(((xx)(xx))((xx)(xx)))(((yy)(yy))((yy)(yy))) \approx (xx)(((yy)(xx))((yy)(xx))) \in Id\mathcal{K}'$.

By definition of hyperidentity, we see that if $s \approx t$ is a hyperidentity in $\mathcal{K}'$, then we can find the others identities in $\mathcal{K}'$ by apply some hypersubstitutions to $s \approx t$ as the following examples:

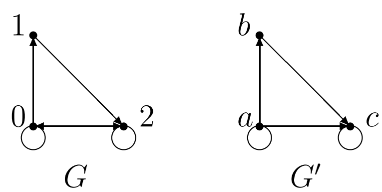
Example 5.4. In Example 5.2 we know that $x((yx)y) \approx (xx)((yx)y)$ is a hyperidentities in $\mathcal{K}'$. We can get other identities in $\mathcal{K}'$ by apply some hypersubstitution to $x((yx)y) \approx (xx)((yx)y)$ as the following:

(a) Apply σ_7 to $x((yx)y) \approx (xx)((yx)y)$, we get
$\hat{\sigma}_7[x((yx)y)] = (((y((xy)x))y)x)((y((xy)x))y)$ and
$\hat{\sigma}_7[(xx)((yx)y)] = (((y((xy)x))y)((xx)x))((y((xy)x))y)$.
Hence $(((y((xy)x))y)x)((y((xy)x))y) \approx (((y((xy)x))y)((xx)x))((y((xy)x))y) \in Id\mathcal{K}'$.

(b) Apply σ_9 to $x((yx)y) \approx (xx)((yx)y)$, we get
$\hat{\sigma}_9[x((yx)y)] = (y((x(yy))(x(yy))))(xx)$ and
$\hat{\sigma}_9[(xx)((yx)y)] = (y((x(yy))(x(yy))))((x(xx))(x(xx)))$.
Hence $(y((x(yy))(x(yy))))(xx) \approx (y((x(yy))(x(yy))))((x(xx))(x(xx))) \in Id\mathcal{K}'$.

Let $G \in \mathcal{K}'$ and $s \approx t \in Id\mathcal{K}'$. We can substitute the subgraph of G which isomorphic to $G(s)$ by $G(t)$ to get new graph G'. We have if G is a diagram of some job, then the result of the job process by G and G' are the same.

Example 5.5. Since $((xx)(y((zx)z)))z \approx ((xx)(y(zz)))z \in Id\mathcal{K}'$. We have the result of the job process by G and G' as below are the same but process by G' is better because it reduce the cost and reduce the time.



Definition 5.1. Let $\mathcal{R}$ be the class of directed graph without multiple edges. The direct product $G = \prod_{i \in I} G_i$ is defined by $V(G) = \prod_{i \in I} V(G_i)$ (cartesian product) and $E(G) = \{(a, b) \in V(G) \times V(G) \mid (a(i), b(i)) \in E(G_i), i \in I\}$.

In [10] it proved that the direct product of the elements of $\mathcal{R}$ belongs to $V_g(\mathcal{R})$. Hence, we give the following example:

Example 5.6. Let G be the $((xx)(y((zx)z)))z$ graph, let $G_1 = G \times G$ and let G' be the $((xx)(y(zz)))z$ graph. Since $((xx)(y((zx)z)))z \approx ((xx)(y(zz)))z \in Id\mathcal{K}'$. We have if G_2 is a graph which obtains from G_1 by substitute every subgraph of G_1 which isomorphic to G by G' (i.e. $G_2 = G' \times G'$), then the result of the job process by G_1 and G_2 are the same but process by G_2 is better because it reduce cost and reduce time.

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