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On Usual and Singular Spectrum in the Fundamental Locally Multiplicative Topological Algebra

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Abstract. Fundamental topological algebras has been introduced to generalize the meaning of locally bounded and locally convex algebras. In this paper we shall study several new results on locally multiplicative fundamental topological algebras which has a property similar to the Banach algebras.

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1. Introduction

The fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending the meaning of locally bounded and locally convex algebras. Some basic theorems as well as are proved on fundamental topological vector space and fundamental topological algebras in [1], [2], [3], [4]. A natural question is to ask for generalizing the basic results on this new class of topological algebras.

The fundamental locally multiplicative topological algebras (abbreviated by FLM) with a property very similar to the normed algebras, is also introduced in [3]. Also topological structure is defined on the algebraic dual space of an FLM algebra to make it a normed space, and some of famous theorems of Banach algebras are extended to complete metrizable FLM algebras.

In this note we state and prove some new results, in particular concerning the usual and singular spectrum and topological zero divisor from Banach algebras to FLM algebras.

2. Definition and related results

Definition 2.1. *A topological linear space A is said to be fundamental one if there exists $b > 1$ such that for every sequence $\{x_n\}$ of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that $\{x_n\}$ is Cauchy.*

Definition 2.2. *A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.*

Definition 2.3. *In fundamental topological algebra, we define:*
$S(A) = \{x \in A : \exists b > 1 \;\; s.t \quad b^n x^n \to 0\}$.

Theorem 2.4 (2)**.** *Let A be a complete metrizable fundamental topological algebra, and $x \in S(A)$, if A possesses a unit element, then $e - x$ is invertible and*

$$(e - x)^{-1} = e + \sum_{n=1}^{\infty} x^n$$

Definition 2.5. *A fundamental topological algebra is called to be locally multiplicative, if there exists a neighbourhood U_0 of zero such that for every neighbourhood V of zero, the sufficiently large powers of U_0 lie in V. We call such an algebra, an FLM algebra.*

Definition 2.6. *Let A be an FLM algebra with unit element e. The set of all invertible elements of A and the set of all boundary points of A will be denoted by $Inv(A)$ and $\partial(A)$ respectively.*

Theorem 2.7 (2)**.** *Let A be a complete metrizable FLM algebra with a unit element . Then $Inv(A)$, is an open subset of A.*

Definition 2.8. *Let A be an FLM algebra with unit element e. An element $a \in A$ will be called a left (right) topological zero divisor if there exists a sequence $\{x_n\}$ in A such that $x_n \not\to 0$ and $ax_n \to 0$ $(x_n a \to 0)$ as $n \to \infty$. A topological zero divisor is both a left and right topological zero divisor. If A is commutative, then every left topological zero divisor or right topological zero divisor is a topological zero divisor.*

Definition 2.9. *Let A be an FLM algebra with unit element e. The usual spectrum and the singular spectrum of $x \in A$ are denoted by $sp(x)$ and $\sigma(x)$*

and defined in the following:
$sp(x) = \{\lambda \in C \ : \ \ x - \lambda e \ \ is\, not\, invertible\}$,
$\sigma(x) = \{\lambda \in C \ : \ \ x - \lambda e \ \ is\, a\, topological\, zero\, divisor\}$.

Theorem 2.10 (2)**.** *Let A be a complete metrizable FLM algebra with a unit element and let $x \in A$. Then, the $sp(x)$ is compact.*

Theorem 2.11 (2)**.** *Let A be a complete metrizable FLM algebra. Then, every multiplicative linear functional is continuous.*

Corollary 2.12 (3)**.** *In a complete metrizable FLM algebra A, if $a \in \partial Inv(A)$, and $a_n \to a$ with $a_n a = a a_n$ then the set $\{a_n^{-1} : \ n \in N\}$ is unbounded.*

3. Usual spectrum and topological zero divisor

On FLM algebras with a property similar to the Banach algebras, some new theorems and results have been obtained before. In this section we continue this process and get some results.

In Banach algebra B for every element $x \in B$, $sp(x)$ is nonvoid and compact [6]. Compactness of $sp(x)$ for complete metrizable FLM algebras with a unit element is proved in [2, 4.4]. This is also true for locally m-convex Q-algebras[9].

Now we would like to prove the following theorem for FLM algebra.

Theorem 3.1. *let A be a complete metrizable FLM algebra with the unit element e and A' be a topological dual space of A. Set*

$$S(A) = \{x \in A \ : \ \exists\, b > 1 \ \ s.t \ \ b^n x^n \to 0\}$$

suppose that either:
(i) A' separates the point of A ; or
(ii) $\ker\varphi \subseteq \sin g(A) \ \ , \ \ \varphi \neq 0 \in A'$
then $sp(x) \neq \emptyset$, $\forall x \in S(A)$.

Proof. Let $sp(x) = \emptyset$ for some $x \in S(A)$. Then for each $z \in C$, $(x - ze)$ is invertible. We define $R_x : \ C \to A$ with

$$R_x(z) = (x - ze)^{-1} \ \ , \ \ \forall z \in \ C$$

by [2, 4.1] we have

$$(e - x)^{-1} = e + \sum_{n=0}^{\infty} x^n$$

suppose that $z > 1$, then there exists some $b > 1$ such that

$$z^{-n}b^n x^n \to 0 \ \ or \quad b^n(\frac{x}{z})^n \to 0$$

In this case we have

$$(e - \frac{x}{z})^{-1} = e + \sum_{n=0}^{\infty} x^n z^{-n}$$

and so

$$(ze - x)^{-1} = z^{-1} + \sum_{n=0}^{\infty} x^n z^{-n-1}$$

on the other hand, from
$[S(A) = b^{-1}U_0 \Rightarrow (S(A))^n = b^{-n}U_0^n \Rightarrow b^n(S(A))^n = U_0^n \Rightarrow for\ \ x \in S(A),\ b^n x^n \to 0, where$ U_0 is the neighbourhood satisfying in the definion of FLM algebra (see 2.5)], we deduced that $S(A)$ is bounded.

Since $\varphi \in A'$ is continuous, therefore φ is bounded on $S(A)$, i.e.

$$\exists M > 0 \ s.t \ |\varphi(x)| \leq M \ , \ \ \forall x \in S(A)$$

also suppose that $b > 1$ and $y \in S(A) = b^{-1}U_0$. Put $x = y^k$ for $k \in N$. Since $(b^k)^n x^n = b^{kn}y^{kn} \to 0$ therefore $|\varphi(x)| \leq M$ i.e. $\left|\varphi(y^k)\right| \leq M$, thus this discussion and continuity of $\varphi \in A'$ implies that we have the following,

$$\varphi(R_x(z)) = \varphi(x - ze)^{-1}$$

and
$\left|\varphi(x - ze)^{-1}\right| = \left|\varphi(\sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}})\right| \leq \left|\sum_{n=0}^{\infty} \frac{\varphi(x^n)}{z^{n+1}}\right| \leq \sum_{n=0}^{\infty} \frac{|\varphi(x^n|}{|z^{n+1}|} \leq \sum_{n=0}^{\infty} \frac{M}{|z^{n+1}|}$
$\leq \frac{M}{1-z} \to 0 \ \ as\ z \to \infty$ hence $\left|\varphi(x - ze)^{-1}\right| \to 0 \ \ as\ z \to \infty$ since $\varphi(R_x(z))$ is an entire function, so $\varphi(R_x(z)) = 0$ by Liouville's theorem. Therefore

$$\varphi(R_x(z)) = \varphi(x - ze)^{-1} = 0$$

In the case (i) since φ separates the point of A, so $(x - ze)^{-1} = 0$. In the case (ii) $(x - ze)^{-1} \in \ker \varphi \subseteq \sin g(A)$, thus the cases of (i) and (ii) lead to contradiction, consequently $sp(x) \neq \emptyset$, $\forall x \in S(A)$

□

In Banach algebra B with the unit element e, every boundary point of $Inv(B)$, the group of invertibale elements is a topological zero divisor [11]. Here we prove a similar result for complete metrizable FLM algebra.

Theorem 3.2. *Suppose A is a commutative complete metrizable FLM algebra with unit element e and $Inv(A)$ the group of invertible elements of A. If $x \in \partial Inv(A)$ then x is a topological zero divisor.*

Proof. If $x \in \partial Inv(A)$, then there exists a sequence $\{x_n\} \subseteq A$ $s.t$ $x_n \to x$. By theorem or corollary [3, 3.5] the set $E = \{x_n^{-1} \ : \ n \in N\}$ is unbounded. So there is a neighbourhood V of zero and a sequence of scalar $r_n \to \infty$ such that no r_nV contains E. Choose $x_n^{-1} \in E$ such that $x_n^{-1} \notin r_nV$. Then no $r_n^{-1}x_n^{-1}$ is in V, so that $\{r_n^{-1}x_n^{-1}\}$ does not converge to zero, i.e. $r_n^{-1}x_n^{-1} \not\to 0$ as $n \to \infty$. By continuity of multiplication in topological algebra, we have $\lim_{n\to\infty} xr_n^{-1}x_n^{-1} = \lim_{n\to\infty} x_nr_n^{-1}x_n^{-1} = \lim_{n\to\infty} r_n^{-1}e = 0$, i.e. $xr_n^{-1}x_n^{-1} \to 0$, consequently x is a left topological zero divisor. Similarly x is a right topological zero divisor, thus x is topological zero divisor.

□

Corollary 3.3. *Let A be a commutative complete metrizable FLM algebra with unit element e. If $\lambda \in \partial sp(x)$ then $x - \lambda e$ is topological zero divisor.*

Proof. By theorem [2, 4.4] $sp(x)$ is compact. Since $\lambda \in \partial sp(x)$ and $sp(x)$ is closed set, therefore there exists $\{\lambda_n\} \subseteq C\backslash sp(x)$ such that $\lambda_n \to \lambda$. Hence $x - \lambda_n e \to x - \lambda e$, it follows that $x - \lambda e \in \partial Inv(A)$ and by (theorem 3.2) $x - \lambda e$ is a topological zero divisor.

□

Corollary 3.4. *(Gelfand-Mazur theorem). Let A be a unital commutative complete metrizable FLM algebra and $\partial sp(x) \neq \emptyset$, $\forall x \in A$. If A has no non-zero topological zero divisor, then $A = Ce$.*

Proof. If $\lambda \in \partial sp(x)$ then by the above corollary $x - \lambda e$ is a topological zero divisor and so our assumption implies that $x - \lambda e = 0$ or $x = \lambda e$, $i.e.$ $A = Ce$.

□

Theorem 3.5. *Suppose A is a commutative complete metrizable FLM algebra with the identity element e and $\{x_n\}$ is sequence of $Inv(A)$ such that $x_n \to x \in A$, then $x \in Inv(A) \Leftrightarrow \{x_n^{-1} : \ n \in N\}$ is bounded.*

Proof. Suppose that $x_n \in Inv(A)$ and $x_n \to x$, $x \in Inv(A)$, since an open set is $G_\delta - set$ and $Inv(A)$ is open set in FLM algebra [2, 4.3]. Therefore $Inv(A)$ is a $G_\delta - set$ and by [7, 2.2.38] inversion is continuous for A, and so $x_n^{-1} \to x^{-1}$. Therefore $\{x_n^{-1} : \ n \in N\}$ is bounded.
Converse of the theorem has been proved in [3, 3.5].

□

4. **Singular spectrum and homomorphism**

In this section we compare the singular spectrum of an FLM algebra element with the usual spectrum. Also some results concerning with the singular spectrum and homomorphism will be studied. First we begin the section with a simple lemma.

Lemma 4.1. *Let A be a complete metrizable FLM algebra with unit element e. Then a topological zero divisor is not invertible.*

Proof. Let x be a topological zero divisor, then there exists a sequence $\{x_n\} \subseteq A$ s.t $x_n \nrightarrow 0$ *and* $xx_n \to 0$. If x is invertible, i.e. $x^{-1}x = xx^{-1} = e$, then we have $x_n = ex_n = x^{-1}xu_n = x^{-1}(xu_n) \to 0$ therefore $x_n \to 0$, a contradiction. □

Theorem 4.2. *Suppose A is a commutative complete metrizable FLM algebra with identity element e. Then the followings hold:*
(i) $\partial sp(x) \subseteq \sigma(x) \subseteq sp(x)$,
(ii) $\sigma(x)$ is compact subset of C.

Proof. (i) Let $\lambda \in \partial sp(x)$, then by (corollary 3.3) $x - \lambda e$ is topological zero divisor and so $\lambda \in \sigma(x)$, i.e. $\partial sp(x) \subseteq \sigma(x)$. Also if $\mu \in \sigma(x)$, then $x - \mu e$ is topological zero divisor and so by (lemma 4.1) $x - \mu e \notin Inv(A)$, therefore $\mu \in sp(x)$. Consequently, $\partial sp(x) \subseteq \sigma(x) \subseteq sp(x)$.

(ii) By theorem [2, 4.4], $sp(x)$ is compact, then by the part of (i), it is enough to show that $\sigma(x)$ is closed. Let $\lambda \in \overline{\sigma(x)}$ then there exists a sequence $\{\lambda_n\} \subseteq \sigma(x)$ s.t $\lambda_n \to \lambda$ *as* $n \to \infty$, *and so* $x - \lambda_n e \to x - \lambda e$ since $x - \lambda_n e$ is topological zero divisor, then $\exists \{x_n\} \subseteq A$ *s.t* $x_n \nrightarrow 0$ and $(x-\lambda_n e)x_n \to 0, i.e.$ $\lim_{n\to\infty}(x - \lambda_n e)x_n = \lim_{n\to\infty}(x - \lambda_n e) \lim_{n\to\infty} x_n = (x - \lambda e) \lim_{n\to\infty} x_n = 0$, therefore $(x-\lambda e)x_n \to 0$, consequently $x-\lambda e$ is topological zero divisor, and so $\lambda \in \sigma(x)$. □

In [12], similar to the following theorem has been studied for usual spectrum on FLM algebras. Here we continue this process about singular spectrum and get some new results.

Theorem 4.3. *Let A and B be two commutative complete metrizable FLM algebras, with unit elements e and e' respectively. If $\theta : A \to B$ is injective continuous homomorphism such that $\theta(e) = e'$, then $\sigma(x) \subseteq \sigma(\theta(x)) \subseteq sp(x)$, $\forall x \in A$.*

Proof. At first we show that $\sigma(x) \subseteq \sigma(\theta(x))$. If $\lambda \in \sigma(x)$ then $x - \lambda e$ is a topological zero divisor, if $x - \lambda e$ is a left topological zero divisor in A, then there exists a sequence $\{z_n\}$ in A with $z_n \nrightarrow 0$ and $(x - \lambda e)z_n \to 0$ *as* $n \to \infty$. Since θ is continuous homomorphism then $(\theta(x) - \lambda e')\theta(z_n) \to 0$ but $\theta(z_n) \nrightarrow 0$ (because if $\theta(z_n) \to 0$ then $\theta(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} \theta(z_n) = \theta(0) = 0$, since θ is injection, therefore $z_n \to 0$, which is a contradiction). Hence $\theta(z_n) \nrightarrow 0$, this implies that $(\theta(x) - \lambda e')$ is a left topological zero divisor, similarly it is a right topological zero divisor, and so it is a topological zero divisor. Consequently $\lambda \in \sigma(\theta(x))$. Therefore we proved $\sigma(x) \subseteq \sigma(\theta(x))$ for all $x \in A$. On the other hand, by using (theorem 4.2) and the relation $sp(\theta(x)) \subseteq sp(x)$ in [12], we conclude that $\sigma(x) \subseteq \sigma(\theta(x)) \subseteq sp(x)\,,\ \forall x \in A$. □

Theorem 4.4. *Let A and B with unit elements e, e' be two commutative complete metrizable FLM algebras and let B be semi-simple. Also let ϕ_A and ϕ_B be carrier spaces of A and B respectively. If $\theta : A \to B$ is injective homomorphism such that $\theta(e) = e'$, then $\sigma(x) \subseteq \sigma(\theta(x)) \subseteq sp(x)\,,\ \forall x \in A$.*

Proof. By [2, 4.5] and similar proof of Silove theorem in [7, 2.3.3] we conclude that θ is continuous. The remainder proof of the theorem follows by the same reasoning in the preceding theorem. □

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