International Mathematical Forum, Vol. 7, 2012, no. 24, 1149 - 1169

Symmetric Positive Systems Applied to Partial Differential Equations

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Abstract. The existence and uniqueness theorems for weak and strong solutions via the symmetric positive systems, are applied to solve the self-adjoint Neumann problem, the non self-adjoint Neumann problem and the non self-adjoint Dirichlet problem for elliptic and hyperbolic equations of the second order.

Mathematics Subject Classification: 35G05, 35G16, 35G35

Keywords: Symmetric positive operator, Symmetric positive systems, Weak solutions, Strong solutions

1. Introduction

The theory of symmetric positive systems was introduced by K. O. Friedrich in [2] to prove the existence and uniqueness theorems for different types of partial differential equations with boundary conditions. Indeed, in [2] it is shown that the wave equation, the heat equation, the Laplace equation, the Tricomi equation, and the ultra-hyperbolic equation can be transformed to a symmetric positive systems.

Also, in [3], the theory of symmetric positive systems has been used to prove the uniqueness and existence for a special ODE, and in [1] the author gives conditions to show that a strong solution of a symmetric positive system as defined in [2] belongs to H_s.

In this paper we show that the following boundary problems of the second order can be reduced to a symmetric positive systems: The self-adjoint Neumann problem, the non self-adjoint Neumann problem and the non self-adjoint Dirichlet problem.

2. Notations and Definitions

In order to define the symmetric positive systems, we will follow the definitions and notations given in [2]. So, for the reader's convenience, in this section we summarize these basic concepts.

Consider a bounded region Ω in $\mathbb{R}^m$. Let k be a positive integer, and for each $\rho = 1, 2, \ldots, m$, each $\lambda = 1, 2, \ldots, k$, and each $\nu = 1, 2, \ldots, k$, let $\lambda^{\rho}_{\lambda\nu} : \overline{\Omega} \to \mathbb{R}$ be a function of class C_1 in $\overline{\Omega}$. Also, for each $\lambda = 1, 2, \ldots, k$, and each $\nu = 1, 2, \ldots, k$, let $\gamma^{\rho}_{\lambda\nu} : \Omega \to \mathbb{R}$ be a continuous function on Ω.

Note. We will consider the following convention: Every time and index is repeated, it will be understood that we will add over this index.

Definition 1. *For any function $u : \Omega \to \mathbb{R}^k$ of class C_1, define the differential operator L as*

$$Lu = 2\alpha^{\rho}\frac{\partial u}{\partial x_{\rho}} + \gamma u, \tag{1}$$

where α^{ρ} and γ are the $k \times k$ matrices defined by

$$\alpha^{\rho} = (\alpha^{\rho}_{\lambda\nu}) \quad \textit{and} \quad \gamma = (\gamma_{\lambda\nu}).$$

Definition 2. *The differential operator L defined in (1) is said to be symmetric positive if*

1) The matrix α^{ρ} is symmetric for any $\rho = 1, 2, \ldots, m$.

2) The symmetric part of the matrix $\xi = \gamma - \dfrac{\partial \alpha^{\rho}}{\partial x_{\rho}}$ is positive definite.

Definition 3. *Consider a function $f : \Omega \to \mathbb{R}^k$ such that $f \in L^2(\mathbb{R})$. The system*

$$\begin{cases} Lu(x) = f(x) & \textit{if} \quad x \in \Omega \\ \beta_{-}(x)u(x) = 0 & \textit{if} \quad x \in \partial\Omega, \end{cases} \tag{2}$$

is said to be symmetric positive if

1) The differential operator L is symmetric positive.

2) For each $x \in \partial\Omega$ there are two $k \times k$ matrices $\beta_+(x)$ and $\beta_-(x)$ such that the matrix $\beta(x) \equiv \eta_\rho(x)\alpha^\rho(x)$ can be written as $\beta(x) = \beta_+(x) + \beta_-(x)$, and where the symmetric part of the matrix $\beta_+(x) - \beta_-(x)$ is non-negative. Note that $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is the normal vector in each point of $\partial\Omega$.

In this case the boundary condition $\beta_-(x)u(x) = 0$ is called semi-admissible for the differential operator L.

Moreover. If the matrices $\beta_+(x)$ and $\beta_-(x)$ satisfy that for each function $u : \Omega \to \mathbb{R}^k$ there are two functions $u_+ : \Omega \to \mathbb{R}^k$ and $u_- : \Omega \to \mathbb{R}^k$ so that $u(x) = u_+(x) + u_-(x)$ and $\beta_+(x)u_-(x) = 0 = \beta_-(x)u_+(x)$, then the condition $\beta_-(x)u(x) = 0$ is called an admissible boundary condition for the differential operator L.

Definition 4. *Consider a function $f : \Omega \to \mathbb{R}^k$ such that $f \in L^2(\mathbb{R})$. We say that the function $u : \Omega \to \mathbb{R}^k$ where $u \in L^2(\mathbb{R})$ is a weak solution of the differential equation $Lu = f$ under the semi-admissible boundary condition $\beta_-u = 0$, if for each function $v \in C^1(\overline{\Omega})$, where $\beta_+^t v = 0$ we have*

$$\int_\Omega u \cdot (L^* v) = \int_\Omega f \cdot v$$

In this case, β_+^t is the transpose of β_+, and where L^* is the adjoint of L given by:

$$L^* = -2\alpha^\rho \frac{\partial}{\partial x_\rho} - \frac{\partial \alpha^\rho}{\partial x_\rho} + \xi^t.$$

Then we have the following result:

Lemma 1. *Every classical solution of (2) is also a weak solution.*

Proof. Consider $v \in C^1(\overline{\Omega})$, where $\beta_+^t v = 0$. Then

$$L^* v = -2\alpha^\rho \frac{\partial v}{\partial x^\rho} - \frac{\partial \alpha^\rho}{\partial x^\rho} v + \xi^t v = -2\alpha^\rho \frac{\partial v}{\partial x^\rho} - 2\frac{\partial \alpha^\rho}{\partial x^\rho} v + \gamma^t v$$

Hence,

$$\int_\Omega (L^* v) \cdot u = \int_\Omega \left(-2\alpha^\rho \frac{\partial v}{\partial x^\rho}\right) u + \int_\Omega \left(-2\frac{\partial \alpha^\rho}{\partial x^\rho} v\right) u + \int_\Omega \gamma^t v u$$

Thus,

$$\begin{aligned}\int_\Omega (L^*v)\cdot u &= \int_\Omega -2\frac{\partial v}{\partial x^\rho}(\alpha^\rho u) + \int_\Omega -2v\left(\frac{\partial \alpha^\rho}{\partial x^\rho}u\right) + \int_\Omega \gamma^t vu\\ &= \int_{\partial\Omega} -2v(\eta_\rho \alpha^\rho)u + \int_\Omega 2v\frac{\partial(\alpha^\rho u)}{\partial x^\rho} + \int_\Omega -2v\left(\frac{\partial \alpha^\rho}{\partial x^\rho}u\right) + \int_\Omega \gamma^t vu\\ &= \int_{\partial\Omega} -2v(\beta_+ + \beta_-)u + \int_\Omega 2v\frac{\partial(\alpha^\rho u)}{\partial x^\rho} + \int_\Omega -2v\left(\frac{\partial \alpha^\rho}{\partial x^\rho}u\right) + \int_\Omega \gamma^t vu\end{aligned}$$

Note that since $\beta_+^t v = 0$ and $\beta_- u = 0$, it follows that

$$\int_{\partial\Omega} -2v(\beta_+ + \beta_-)u = \int_{\partial\Omega} -2(\beta_+^t v)u + \int_{\partial\Omega} -2v(\beta_- u) = 0$$

Thus,

$$\begin{aligned}\int_\Omega (L^*v)u &= \int_\Omega 2v\frac{\partial(\alpha^\rho u)}{\partial x^\rho} + \int_\Omega -2v\left(\frac{\partial \alpha^\rho}{\partial x^\rho}u\right) + \int_\Omega \gamma^t vu\\ &= \int_\Omega 2v\alpha^\rho\frac{\partial u}{\partial x^\rho} + \int_\Omega 2v\frac{\partial \alpha^\rho}{\partial x^\rho}u + \int_\Omega -2v\left(\frac{\partial \alpha^\rho}{\partial x^\rho}u\right) + \int_\Omega \gamma^t vu\\ &= \int_\Omega 2v\alpha^\rho\frac{\partial u}{\partial x^\rho} + \int_\Omega \gamma^t vu = \int_\Omega 2v\alpha^\rho\frac{\partial u}{\partial x^\rho} + \int_\Omega v\gamma u\\ &= \int_\Omega v\left(2\alpha^\rho\frac{\partial u}{\partial x^\rho} + \gamma u\right) = \int_\Omega v\cdot(Lu) = \int_\Omega v\cdot f\end{aligned}$$

□

The following two theorems related with existence and uniqueness for weak solutions are given in [2].

Theorem 1. *(Existence) If the system (2) is symmetric positive, then for any $f \in L^2(\Omega)$, this system has a weak solution*

Theorem 2. *(Uniqueness) If the system (2) is symmetric positive, then for any $f \in L^2(\Omega)$, this system has at most one weak solution in $C_1(\Omega)$.*

Note. The proofs of the last two Theorems are based on the following classical inequality: There is $C > 0$ such that if u is in the domain of L and $\beta_- u = 0$, then

$$(Lu, u) \geq C\|u\|^2 \tag{3}$$

We will show that the following boundary problems: The non self-adjoint Neumann problem, the self-adjoint Neumann problem and the non self-adjoint

Dirichlet problem, can be written as symmetric positive systems, so that from theorems 1 and 2, there is one and only one weak solution.

3. The non self-adjoint Neumann problem

Theorem 3. *Consider a bounded region Ω in $\mathbb{R}^m$, a function $h : \Omega \to \mathbb{R}$ so that $h \in L^2(\Omega)$, and consider also m functions $A_1, A_2, \ldots, A_m : \overline{\Omega} \to \mathbb{R}$, Then there is one and only one weak solution $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies*

$$\begin{cases} \displaystyle\sum_{j=1}^{m} \left(\partial_j^2 \phi + A_j \partial_j \phi\right) = h(x) \quad \textit{if} \quad x \in \Omega \\ \\ \dfrac{\partial \phi(x)}{\partial \eta} = 0 \quad \textit{if} \quad x \in \partial\Omega, \end{cases} \tag{4}$$

where $\dfrac{\partial \phi(x)}{\partial \eta}$ represents the normal derivative of ϕ at x.

Proof. Consider the special case when $\Omega = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$, and consider the $m+1$ functions $u_j = \partial_j \phi$ for $j = 0, 1, 2, \ldots, m$. Then the first equation in (4) can be written as $Qu = H$, where

1)

$$Q = \begin{pmatrix} Q_{11} & Q_{1m} \\ Q_{1m}^t & I_{mm} \end{pmatrix}$$

is a $(m+1) \times (m+1)$ matrix so that Q_{11} is the 1×1 matrix whose entry is $-\sum_{j=1}^{m} A_j \partial_j$, the matrix Q_{1m} is the $1 \times m$ matrix whose entries are $-\partial_j$ for $j = 1, 2, \ldots, m$, and I_{mm} is the $m \times m$ identity matrix.

2) The matrix u is the $(m+1) \times 1$ matrix whose entries are u_j for $j = 0, 1, \ldots, m$, and

3) H is the $(m+1) \times 1$ matrix whose entries are $0's$ except the first entry, which is $-h$.

On the other hand, note that the matrix Q can be written as

$$Q = \sum_{j=1}^{m} \alpha^j \partial_j + \gamma,$$

where α^j and γ are the $(m+1) \times (m+1)$ matrices given by

$$\alpha^j = \begin{pmatrix} \alpha_{11,j} & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix}, \tag{5}$$

and where for the matrix α^j, we have:

$\alpha_{11,j}$ is the 1×1 matrix whose entry is $-A_j$,

$\alpha_{1m,j}$ is the $1 \times m$ zero matrix except in the entry j which is -1,

$\mathcal{O}_{mm}$ is the $m \times m$ zero matrix.

For the matrix γ, we have:

$\mathcal{O}_{11}$ is the 1×1 zero matrix, $\mathcal{O}_{1m}$ is the $1 \times m$ zero matrix, and I_{mm} is the $m \times m$ identity matrix.

Then we have

a) The matrices α^j are symmetric for each $j = 1, 2, \ldots, m$

b) For $\xi = \gamma - \sum_{j=1}^{m} \partial_j \alpha^j$, we get

$$\xi = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix} - \sum_{j=1}^{m} \partial_j \begin{pmatrix} -A_j & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} \partial_j A_j & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix}.$$

In this case, $\xi^t = \xi$. So, the symmetric part of ξ is again ξ. Thus, the symmetric part of ξ is positive definite if

$$\sum_{j=1}^{m} \partial_j A_j > 0. \tag{6}$$

Now, let us prove that there exists two matrices β_+ and β_- so that the matrix $\beta = \eta_j \alpha^j$ can be written as $\beta = \beta_+ + \beta_-$ so that the boundary condition $\beta_- u = 0$ is admissible.

For this purpose, for the bounded region Ω, let us consider the two possibilities for the unit normal vector: η_j and $-\eta_j$, where $\eta_j = (\eta_{j1}, \eta_{j2}, \ldots, 1, \ldots, \eta_{jm})$, and where 1 is in jj^{th} entry, with $j = 1, 2, \ldots, m$. Hence,

$$\beta = \eta_j \alpha^j = \eta_{j1}\alpha^1 + \eta_{j2}\alpha^2 + \cdots + \eta_{jj}\alpha^j + \cdots + \eta_{jm}\alpha^m$$
$$= 0\alpha^1 + 0\alpha^2 + \cdots + 1\alpha^j + \cdots + 0\alpha^m = \alpha^j = \begin{pmatrix} -A_j & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} -A_j & \mathcal{O}_{1m} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right]$$
$$= \frac{1}{2}\left[\begin{pmatrix} -A_j & -\alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} + \begin{pmatrix} -A_j & \alpha_{1m,j} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}\right] = \begin{pmatrix} -A_j & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix},$$

and this matrix is non-negative if

$$A_j \leq 0. \tag{7}$$

Now, for

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$ we note that u can be written as $u = u_+ + u_-$, where

$$u_+ = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} -u_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_j = \frac{\partial \phi}{\partial x_j} = 0. \tag{8}$$

That is, in the direction $\eta_j = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is admissible.

On the other hand, for the unit normal vector $-\eta_j = (0, 0, \ldots, -1, \ldots, 0)$, where -1 is in jj^{th} entry, we have

$$\beta = 0\alpha^1 + 0\alpha^2 + \cdots - 1\alpha^j + \cdots + 0\alpha^m = -\alpha^j = \begin{pmatrix} A_j & -\alpha_{1m,j} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} A_j & \mathcal{O}_{1m} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} \mathcal{O}_{11} & -\alpha_{1m,j} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right]$$
$$= \frac{1}{2}\left[\begin{pmatrix} -A_j & \alpha_{1m,j} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} + \begin{pmatrix} -A_j & -\alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}\right] = \begin{pmatrix} A_j & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix},$$

and this matrix is non-negative if

$$A_j \geq 0. \tag{9}$$

Now, for

$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$, we note that u can be written as $u = u_+ + u_-$, where

$$u_+ = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} u_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_j = \frac{\partial \phi}{\partial x_j} = 0. \tag{10}$$

That is, in the direction $-\eta_j = (0, 0, \ldots, -1, \ldots, 0)$, where -1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is also admissible.

Hence, problem (4) is equivalent to a symmetric positive system if
$A_j \leq 0$ in $\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_j = 1 \quad \text{and} \quad |x_r| \leq 1 \quad \text{if} \quad r \neq j\}$,
$A_j \geq 0$ in $\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_j = -1 \quad \text{and} \quad |x_r| \leq 1 \quad \text{if} \quad r \neq j\}$,
and (6) is satisfied. That is, if $\sum_{j=1}^{n} \partial_j A_j > 0$.

In the case that Ω is any bounded region in $\mathbb{R}^m$, then problem (4) is equivalent to a symmetric positive system if

$$\langle A_1(x), A_2(x), \ldots, A_m(x) \rangle \cdot \eta_j(x) \leq 0 \quad \text{for any} \quad x \in \partial\Omega, \text{ and } \sum_{j=1}^{n} \partial_j A_j > 0.$$

Therefore, from Theorems 1 and 2, there is a unique weak solution $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies problem (4). □

4. The self-adjoint Neumann problem

Theorem 4. *Consider a bounded region Ω in $\mathbb{R}^m$, a function $h : \Omega \to \mathbb{R}$ so that $h \in L^2(\Omega)$, and consider also a constant C, Then there is one and only one weak solution $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies*

$$\begin{cases} \displaystyle\sum_{j=1}^{m} \partial_j^2 \phi + C\phi = h(x) \quad \textit{if} \quad x \in \Omega \\ \\ \dfrac{\partial \phi(x)}{\partial \eta} = 0 \quad \textit{if} \quad x \in \partial\Omega, \end{cases} \tag{11}$$

where $\dfrac{\partial \phi(x)}{\partial \eta}$ *represents the normal derivative of* ϕ *at* x.

Proof. Consider the special case when $\Omega = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$, and consider the $m+1$ functions $u_j = \partial_j \phi$ for $j = 0, 1, 2, \ldots, m$. Then the first equation in (11) can be written as $Q'u = H$, where,

1)

$$Q' = \begin{pmatrix} Q'_{11} & Q_{1m} \\ Q^t_{1m} & I_{mm} \end{pmatrix}$$

is a $(m+1) \times (m+1)$ matrix so that Q'_{11} is the 1×1 matrix whose entry is $-C$, the matrix Q_{1m} is the $1 \times m$ matrix whose entries are $-\partial_j$ for $j = 1, 2, \ldots, m$, and I_{mm} is the $m \times m$ identity matrix.

2) The matrix u is the $(m+1) \times 1$ matrix whose entries are u_j for $j = 0, 1, \ldots, m$, and

3) H is the $(m+1) \times 1$ matrix whose entries are $0's$ except the first entry, which is $-h$.

Clearly the matrix Q' can be written as $Q' = \sum_{j=1}^m \alpha^j \partial_j + \gamma$, where α^j and γ are the $(m+1) \times (m+1)$ matrices given by

$$\alpha^j = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \gamma_{11} & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix}, \tag{12}$$

and where for the matrix α^j,

$\mathcal{O}_{11}$ is the 1×1 zero matrix,

$\alpha_{1m,j}$ is the $1 \times m$ zero matrix except in the entry j which is -1, and

$\mathcal{O}_{mm}$ is the $m \times m$ zero matrix.

For the matrix γ,

γ_{11} is the 1×1 matrix whose entry is $-C$, $\mathcal{O}_{1m}$ is the $1 \times m$ zero matrix, and I_{mm} is the $m \times m$ identity matrix.

Then we have

a) The matrices α^j are symmetric for each $j = 1, 2, \ldots, m$

b) For $\xi = \gamma - \sum_{j=1}^{m} \partial_j \alpha^j$, we get

$$\xi = \begin{pmatrix} \gamma_{11} & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix} - \sum_{j=1}^{m} \partial_j \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix} = \begin{pmatrix} -C & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & I_{mm} \end{pmatrix}.$$

In this case, $\xi^t = \xi$. So, the symmetric part of ξ is again ξ. Thus, the symmetric part of ξ is positive definite if

$$C < 0. \tag{13}$$

Now, let us prove that there exists two matrices β_+ and β_- so that the matrix $\beta = \eta_j \alpha^j$ can be written as $\beta = \beta_+ + \beta_-$ so that the boundary condition $\beta_- u = 0$ is admissible.

For this purpose, for the bounded region Ω, let us consider the two possibilities for the unit normal vector: η_j and $-\eta_j$, where $\eta_j = (0, 0, \ldots, 1, \ldots, 0)$, and where 1 is in jj^{th} entry. Hence,

$$\beta = 0\alpha^1 + 0\alpha^2 + \cdots + 1\alpha^j + \cdots + 0\alpha^m = \alpha^j = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ \alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right] = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix} \tag{14}$$

and this matrix is non-negative.

Now, for

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad \text{we note that } u \text{ can be written as } u = u_+ + u_-, \text{ where}$$

$$u_+ = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} -u_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_j = \frac{\partial \phi}{\partial x_j} = 0. \tag{15}$$

That is, in the direction $\eta_j = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is admissible.

In the direction $-\eta_j = (0, 0, \ldots, -1, \ldots, 0)$, where -1 is in jj^{th} entry,

$$\beta = 0\alpha^1 + 0\alpha^2 + \cdots - 1\alpha^j + \cdots + 0\alpha^m = -\alpha^j = \begin{pmatrix} \mathcal{O}_{11} & -\alpha_{1m,j} \\ -\alpha_{m1,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} \mathcal{O}_{11} & -\alpha_{1m,j} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right] = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{O}_{mm} \end{pmatrix} \tag{16}$$

and this matrix is non-negative.

Now, for

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad \text{we note that } u \text{ can be written as } u = u_+ + u_-, \text{ where}$$

$$u_+ = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} -u_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_j = \frac{\partial \phi}{\partial x_j} = 0. \tag{17}$$

That is, in the direction $-\eta_j = (0, 0, \ldots, -1, \ldots, 0)$, where -1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is also admissible.

Hence, problem (11) is equivalent to a symmetric positive system if $C < 0$.

In the case that Ω is any bounded region in $\mathbb{R}^m$, then problem (11) is equivalent to a symmetric positive system if $C < 0$.

Therefore, from Theorems 1 and 2, there is a unique weak solution $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies problem (11). $\square$

5. The non self-adjoint Dirichlet problem

Theorem 5. *Consider a bounded region Ω in $\mathbb{R}^m$, a function $h : \Omega \to \mathbb{R}$ so that $h \in L^2(\Omega)$, and consider also m functions $A_1, A_2, \dots, A_m : \overline{\Omega} \to \mathbb{R}$, Then there is one and only one weak solution $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies*

$$\begin{cases} \sum_{j=1}^{m} \left(\partial_j^2 \phi + A_j \partial_j \phi\right) = h(x) \quad \text{if} \quad x \in \Omega \\ \\ \phi(x) = 0 \quad \text{if} \quad x \in \partial\Omega. \end{cases} \tag{18}$$

Proof. Consider the special case when $\Omega = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$, and consider the $m+1$ functions $u_j = \partial_j \phi$ for $j = 0, 1, 2, \dots, m$. Then the first equation in (18) is equivalent to the first equation in (4). So, we need to change only the decomposition of the matrix β for the two directions η_j and $-\eta_j$.

For the normal vector $\eta_j = (0, 0, \dots, 1, \dots, 0)$, where 1 is in jj^{th} entry, we have

$$\beta = 0\alpha^1 + 0\alpha^2 + \cdots + 1\alpha^j + \cdots + 0\alpha^m = \alpha^j = \begin{pmatrix} -A_j & \alpha_{1m,j} \\ \alpha_{1m,j}^t & \mathcal{O}_{mm} \end{pmatrix}. \tag{19}$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,j} \\ \mathcal{O}_{1m}^t & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} -A_j & \mathcal{O}_{1m} \\ \alpha_{1m,j}^t & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right] = \begin{pmatrix} A_j & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{O}_{mm} \end{pmatrix},$$

and this matrix is non-negative if

$$A_j \geq 0. \tag{20}$$

Now, for

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \quad \text{we note that } u \text{ can be written as } u = u_+ + u_-, \text{ where}$$

$$u_+ = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} -A_j u_0 \\ u_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_0 = \phi = 0. \tag{21}$$

That is, in the direction $\eta_j = (0, 0, \dots, 1, \dots, 0)$, where 1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is admissible.

On the other hand, for the normal vector $-\eta = (0, 0, \dots, -1, \dots, 0)$, where -1 is in jj^{th} entry, we have

$$\beta = 0\alpha^1 + 0\alpha^2 + \cdots - 1\alpha^j + \cdots + 0\alpha^m = -\alpha^j = \begin{pmatrix} A_j & -\alpha_{1m,j} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

This matrix can be written as the sum of two matrices β_+ and β_-, where

$$\beta_+ = \begin{pmatrix} \mathcal{O}_{11} & -\alpha_{1m,j} \\ \mathcal{O}^t_{1m} & \mathcal{O}_{mm} \end{pmatrix}, \quad \text{and} \quad \beta_- = \begin{pmatrix} A_j & \mathcal{O}_{1m} \\ -\alpha^t_{1m,j} & \mathcal{O}_{mm} \end{pmatrix}.$$

Hence, the symmetric part of $\beta_+ - \beta_-$ is

$$\frac{1}{2}\left[(\beta_+ - \beta_-) + (\beta_+ - \beta_-)^t\right] = \begin{pmatrix} -A_j & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{O}_{mm} \end{pmatrix},$$

and this matrix is non-negative if

$$A_j \leq 0. \tag{22}$$

Now, for

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$ we note that u can be written as $u = u_+ + u_-$, where

$$u_+ = \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad \text{and} \quad u_- = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then we have

$$\beta_- u_+ = 0 = \beta_+ u_-,$$

and

$$\beta_- u = \begin{pmatrix} A_j u_0 \\ -u_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if} \quad u_0 = \phi = 0. \tag{23}$$

That is, in the direction $-\eta_j = (0, 0, \dots, -1, \dots, 0)$, where -1 is in jj^{th} entry, the boundary condition $\beta_- u = 0$ is also admissible.

6. Strong Solutions

In this section we will study the existence and uniqueness for the self-adjoint Neumann problem, the non self-adjoint Neumann problem and the non self-adjoint Dirichlet problem for elliptic and hyperbolic equations of the second order.

Definition 5. *We say that $u \in L^2(\Omega)$ is a strong solution of the equation $Lu = f$, where f is in $L^2(\Omega)$ if there is a sequence of functions $u^\nu \in C^1$ where $\|u^\nu - u\|_{L^2} \to 0$ as $\nu \to \infty$ such that $\|Lu^\nu - f\|_{L^2} \to 0$ as $\nu \to \infty$ and $\beta_- u^\nu = 0$.*

Note. In this case, we will assume that Ω is a region whose boundary is a manifold with edges. That is, we will suppose that there are set $U_1, U_2, \dots, U_n$ in $\mathbb{R}^m$ such that

$$\Omega = \bigcup_{i=1}^{n} U_i, \quad \overline{\Omega} = \bigcup_{i=1}^{n} \overline{U_i},$$

and each U_i satisfies one of the following properties:

1) $\overline{U_i} \subset \Omega$

2) $\overline{U_i} \cap \partial\Omega \neq \emptyset$, and there is a diffeomorphism

$$\phi_i : \overline{U_i} \to \{\, (x_1, x_2, \dots, x_m) \mid x_1^2 + x_2^2 + \cdots + x_m^2 \leq 1 \quad \text{and} \quad x_m \leq 0\}$$

such that $\phi_i(z) = (x_1, x_2, \dots, x_{m-1}, 0)$ if $z \in \partial\Omega$

3) $\overline{U_i} \cap \partial\Omega \neq \emptyset$, and there is a diffeomorphism

$$\phi_i : \overline{U_i} \to \{\, (x_1, \dots, x_m) \mid x_1^2 + \cdots + x_m^2 \leq 1 \quad \text{and} \quad x_1 \leq 0, \dots, x_m \leq 0\},$$

and either
$\phi_i(z) = (x_1, x_2, \dots, x_{m-1}, 0)$, or
$\phi_i(z) = (x_1, x_2, \dots, 0, x_m)$ or
$\vdots$
$\phi_i(z) = (0, x_2, \dots, x_m)$ if $z \in \partial\Omega$

The sets U_i will be called patches, and if U_i satisfies 1), the set U_i will be called an interior patch.

Then we have the following Theorem.

Theorem 6. *For each $f \in L^2(\Omega)$ the system (2) has a strong solution if the boundary of Ω is a manifold with edges and for each non interior patch there is a set of operators of first order $D_\sigma, \sigma = 0, 1, \ldots, m$ with*

$$D_\sigma = d_\sigma^\tau \frac{\partial}{\partial x_\tau} + d_\sigma, \quad \tau = 1, 2, \ldots, m,$$

where the numbers d_σ^τ and the matrices d_σ are functions in C^1 that satisfy the following conditions:

1) If $x \in \partial\Omega$, then $d_\sigma^\tau(x)\eta_\tau(x) = 0$

2) Each operator $d^\tau \frac{\partial}{\partial x_\tau} + d$ is linear combination of the operators D_σ with C coefficients and where d^τ and d are in C so that $d^m = 0$ in $\partial\Omega$

3) $D_\sigma L - LD_\sigma$ is a linear combination of the operators D_τ and L. That is, $D_\sigma L - LD_\sigma = p_\sigma^\tau D_\tau + t_\sigma L$, where p_σ^τ are matrices in C and t_τ are matrices in C^1

4) $D_\sigma\beta_- - \beta_- D_\sigma$ is a linear combination of the operators D_τ and β_- .That is,

$$D_\sigma\beta_- - \beta_- D_\sigma = q_\sigma^\tau D_\tau + t_\sigma^{\partial\Omega}\beta_- \,,$$

where

$$t_\sigma^{\partial\Omega} = t_\sigma + \frac{\partial}{\partial x^m} d_\sigma^m$$

5) $\underset{1}{\nu} + \underset{1}{\nu'} \geq 0$, where $\underset{1}{\nu}\underset{1}{u} = \{\nu u_\sigma + q_\sigma^\tau u_\tau\}$ for a given compose system $\underset{1}{\nu} = \{u_\sigma\}$

Proof. See [2]. □

Remark 1. *The operators D_σ, where $\sigma = 0, 1, \ldots, m$ exist when there exist matrices σ_λ and τ_λ in C such that*

$$\frac{\partial \alpha^m}{\partial x_\lambda} = \tau_\lambda \alpha^m + \alpha^m \sigma_\lambda, \quad \lambda = 1, 2, \ldots, m-1$$

Remark 2. *We have the following choices for σ_λ and τ_λ:*

1) If the matrix α^m is non-singular, we can take

$$\sigma_\lambda = (\alpha^m)^{-1} \frac{\partial \alpha^m}{\partial x_\lambda}, \quad \textit{and} \quad \tau_\lambda = 0$$

2) If the matrix α^m is singular, then we ask for the matrices α^m in different points of Ω to be similar. That is, we need the existence of a non-singular matrix $W(x)$ in C^1 such that $\alpha^m(x) = W(x)\alpha^m(x_0)W^t(x)$ so that we can take

$$\sigma_\lambda = \tau_\lambda^t, \quad \textit{and} \quad \tau_\lambda = \frac{\partial W}{\partial x_\lambda} W^{-1}$$

3) In the case that α^m is a constant matrix, we can take

$$\sigma_\lambda = 0 = \tau_\lambda$$

Theorem 7. *If $\beta\_u = 0$ is a semi-admissible boundary condition to the system (2), then each strong solution to (2) is unique.*

Proof. Suppose that $u, v \in L^2(\Omega)$ are strong solutions to the system (2), then there exists sequences (u^ν) and (v^ν) where $\|u^\nu - u\|_{L^2} \to 0$ as $\nu \to \infty$ and $\|v^\nu - v\|_{L^2} \to 0$ as $\nu \to \infty$ such that $\|Lu^\nu - f\|_{L^2} \to 0$ as $\nu \to \infty$ and $\|Lv^\nu - f\|_{L^2} \to 0$ as $\nu \to \infty$.

On the other hand, for $C > 0$, it comes from (3)

$$C\|u^\nu - v^\nu\| \leq (L(u^\nu - v^\nu), u^\nu - v^\nu) \leq \|L(u^\nu - v^\nu)\| \, \|u^\nu - v^\nu\|$$

But since

$$\|L(u^\nu - v^\nu)\| = \|(Lu^\nu - f) + (Lv^\nu - f)\| \to 0 \quad \text{as} \quad \nu \to \infty,$$

it follows that $\|u^\nu - v^\nu\| \to 0$ as $\nu \to \infty$. Thus, $u = v$. □

7. Examples

7.1. **Consider the non-self adjoint Neumann problem given in (4).** According to (5), the matrix α^m is the $(m \times 1) \times (m \times 1)$ matrix given by:

$$\alpha^m = \begin{pmatrix} -A_m & \alpha_{1m,m} \\ \alpha^t_{1m,m} & \mathcal{O}_{mm} \end{pmatrix},$$

which is a singular matrix. Then from 2) of Remark 2, the problem (4) has a strong solution if there is a non-singular matrix $W(x)$ with $x \in \mathbb{R}^m$ so that

$$\alpha^m(x) = W(x)\alpha(x_0)W^t(x).$$

For this, consider

$$w(x) = \sqrt{\frac{A_m(x)}{A_m(x_0)}},$$

where $A_m(x) > 0$ or $A_m(x_0) < 0$ for any $x \in \overline{\Omega} \subset \mathbb{R}^m$ and $A_m(x_0) \neq 0$. Then take

$$W(x) = \begin{pmatrix} w(x) & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{I}_{mm} \end{pmatrix},$$

where $\mathcal{I}_{mm}$ is the $m \times m$ identity matrix except that the $mm - entry$ is $\dfrac{1}{w(x)}$.

Hence,

$$W(x)\alpha(x_0)W^t(x) = \begin{pmatrix} w(x) & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{I}_{mm} \end{pmatrix} \begin{pmatrix} -A_m(x_0) & \alpha_{1m,m} \\ \alpha_{1m,m}^t & \mathcal{O}_{mm} \end{pmatrix} \begin{pmatrix} w(x) & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{I}_{mm} \end{pmatrix}$$

$= \alpha^m(x)$.

Therefore, we can take

$$\tau_\lambda = \frac{\partial W}{\partial x_\lambda} W^{-1} = \begin{pmatrix} \frac{\partial}{\partial x^\lambda} w(x) & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \Theta_{mm} \end{pmatrix} \begin{pmatrix} \frac{1}{w(x)} & \mathcal{O}_{1m} \\ \mathcal{O}_{1m}^t & \mathcal{I}'_{mm} \end{pmatrix},$$

where Θ_{mm} is the zero $m \times m$ matrix except that the $mm - entry$ is $\dfrac{\partial}{\partial x_\lambda} \dfrac{1}{w(x)}$, and $\mathcal{I}'_{mm}$ is the identity $m \times m$ matrix whose $mm - entry$ is $w(x)$, and $\sigma_\lambda = \tau_\lambda^t = \tau_\lambda$. Therefore,

$$\frac{\partial \alpha^m}{\partial x_\lambda} = \tau_\lambda \alpha^m + \alpha^m \sigma_\lambda, \quad \lambda = 1, 2, \ldots, m-1.$$

The according to Theorems 6 and 7, the problem (4) has a unique strong solution.

7.2. **Consider the self-adjoint Neumann problem given in (11).** Note that from (12), the matrix $\alpha^m m$ is the $(m \times 1) \times (m \times 1)$ matrix given by

$$\alpha^m = \begin{pmatrix} \mathcal{O}_{11} & \alpha_{1m,m} \\ \alpha_{1m,m}^t & \mathcal{O}_{mm} \end{pmatrix},$$

which is a matrix with constant coefficients. Then from 3) of Remark 2, the problem (11) has a strong solution if $\sigma_\lambda = 0 = \tau_\lambda$ so that

$$\frac{\partial \alpha^m}{\partial x_\lambda} = \tau_\lambda \alpha^m + \alpha^m \sigma_\lambda, \quad \lambda = 1, 2, \ldots, m-1.$$

That is, according to Theorems 6 and 7, the problem (11) has a unique strong solution $\phi : \overline{\Omega} \to \mathbb{R}$ for a given $h : \Omega \to \mathbb{R}$, where $h \in L^2(\Omega)$.

7.3. **Consider the non-self adjoint Dirichlet problem given in (18).** According to (19), the matrix α_m is the $(m \times 1) \times (m \times 1)$ given by

$$\alpha_m = \begin{pmatrix} -A_m & \alpha_{1m,m} \\ \alpha^t_{1m,m} & \mathcal{O}_{mm} \end{pmatrix}.$$

So, like the problem (4), the problem (18) has a unique strong solution $\phi : \overline{\Omega} \to \mathbb{R}$ for a given $h : \Omega \to \mathbb{R}$, where $h \in L^2(\Omega)$.

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Received: October, 2011