International Mathematical Forum, Vol. 7, 2012, no. 21, 1033 - 1037

Generalized $\mathcal{N}$-Fuzzy Ideals in Semigroups

S. Lekkoksung

Rajamangala University of Technology Isan
Khon Kaen Campus, Thailand
Lekkoksung_somsak@hotmail.com

Abstract

In this paper, the concepts of (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal and (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal were introduced which can be regarded as a generalization of conmen correspondence concepts, and some properties of (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal and (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal were discussed.

Mathematics Subject Classification: 06F35, 08A72

Keywords: (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal, (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal, cut set

1 Introduction

The concept of fuzzy set, introduced by Zadeh [7], was applied to the theory of groups by Rosenfeld [6]. Since then, many scholars have studied the theorems of fuzzy subring and various fuzzy ideals. Kuroki studied fuzzy semigroup and fuzzy bi-ideal in a semigroup [1-5]. In this paper, the concepts of (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal and (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal were introduced which can be regarded as a generalization of conmen correspondence concepts, and some properties of (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal and (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal were discussed.

2 Preliminary Notes

A semigroup is more general algebraic system than the groups, and is widely applied in computer sciences. In the following statement, S will stand for a semigroup and λ and μ denote two constants such that $0 \leq \mu < \lambda \leq 1$.

By a subsemigroup T of S we mean a nonempty subset of S such that $TT \subseteq T$. A nonempty subset T of S is called a left ideal (right ideal) of S if $ST \subseteq T(TS \subseteq T)$. A nonempty subset T of S is called an ideal of S if it is both a left ideal and a right ideal of S. A nonempty subset T of S is called a generalized bi-ideal of S if $TST \subseteq T$. If the subset T of S is both a generalized bi-ideal and a subsemigroup of S, then T is called a bi-ideal of S.

A subsemigroup T of S is called an inner ideal if $STS \subseteq T$. A nonempty subset Q of S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$.

Let X be a nonempty set. By an $\mathcal{N}$-fuzzy subset A of X we mean a mapping from X to the closed interval [-1,0]. If A is an $\mathcal{N}$-fuzzy subset of X, then the cut set A_t and open cut set $A_{<t>}$ were defined as:

$$A_t := \{x \in X \mid A(x) \leq t\}, A_{<t>} := \{x \in X \mid A(x) < t\}.$$

Definition 2.1 *Let A be an $\mathcal{N}$-fuzzy subset of S. Then A is called a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S if for all $x, y \in S$,*

$$A(xy) \wedge \lambda \leq A(x) \vee A(y) \vee \mu.$$

Definition 2.2 *Let A be an $\mathcal{N}$-fuzzy subset of S. Then A is called a (λ, μ)-generalized $\mathcal{N}$-fuzzy bi-ideal of S if for all $x, y, z \in S$,*

$$A(xyz) \wedge \lambda \leq A(x) \vee A(z) \vee \mu.$$

If A is both a (λ, μ)-generalized $\mathcal{N}$-fuzzy bi-ideal and an anti (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S, then A is called a *(λ, μ)-$\mathcal{N}$-fuzzy bi-ideal* of S

Definition 2.3 *Let A be a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S. Then A is called a (λ, μ)-$\mathcal{N}$-fuzzy inner ideal is for all $x, y, z \in S$,*

$$A(xyz) \wedge \lambda \leq A(y) \vee \mu.$$

Let A and B two $\mathcal{N}$-fuzzy subsets of S. Then the $\mathcal{N}$-fuzzy subset AB is defined as follows: if $x \in S$ can be expressed as $x = x_1 x_2$, then $AB(x) := \inf\{A(x_1) \vee B(x_2) \mid x_1, x_2 \in S\}$. Otherwise $AB(x) = 0$.

Definition 2.4 *Let A be a $\mathcal{N}$-fuzzy subset of S. If for all $x \in S$, $A(x) \wedge \lambda \leq (A1_S)(x) \vee (1_S A)(x) \vee \mu$, then A is called a (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S, where $1_S : S \to \{-1, 0\}$ is the characteristic $\mathcal{N}$-function of S defined by*

$$1_S(x) := \begin{cases} -1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

3 Main Results

Theorem 3.1 *Let A be a (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S. Then A is both a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup and a (λ, μ)-genelarized $\mathcal{N}$-fuzzy bi-ideal of S and hence A is a (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal of S.*

Proof. For all $x, y, z \in S$, we have

$$\begin{aligned} A(xy) \wedge \lambda &\leq (A1_S)(xy) \vee (1_S A)(xy) \vee \mu \\ &\leq A(x) \vee 1_S(y) \vee 1_S(x) \vee A(y) \vee \mu \\ &= A(x) \vee A(y) \vee \mu. \\ A(xyz) \wedge \lambda &\leq (A1_S)(xyz) \vee (1_S A)(xyz) \vee \mu \\ &\leq A(x) \vee 1_S(yz) \vee 1_S(xy) \vee A(z) \vee \mu \\ &= A(x) \vee A(z) \vee \mu. \end{aligned}$$

Hence A is both a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup and a (λ, μ)- generalized $\mathcal{N}$-fuzzy bi-ideal of S and hence A is a (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal bi-ideal of S. ■

Theorem 3.2 *Let T be a nonempty subset of S and let A be an $\mathcal{N}$-fuzzy subset of S such that*

$$A(x) \begin{cases} \geq \lambda & \text{if } x \notin T, \\ \leq \mu & \text{if } x \in T, \end{cases}$$

for all $x \in S$. Then
(1) *A is a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S if T is a subsemigroup of S.*
(2) *A is a (λ, μ)-$\mathcal{N}$-fuzzy bi-ideal of S if T is a bi-ideal of S.*
(3) *A is a (λ, μ)-$\mathcal{N}$-fuzzy inner ideal of S if T is an inner ideal of S.*
(4) *A is a (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S if T is a quasi-ideal of S.*

Proof. We only prove (1) and (4).
(1) Let T be a subsemigroup of S. For all $x, y \in S$, if $x, y \in T$, then $xy \in T$. So we have $A(xy) \leq \mu$ and $A(xy) \wedge \lambda \leq \mu = A(x) \vee A(y) \vee \mu$. If $x \notin T$ or $y \notin T$, then $A(x) \vee A(y) \vee \mu \geq \lambda$. It follows that $A(xy) \wedge \lambda \leq \lambda \leq A(x) \vee A(y) \vee \mu$. So A is a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S.

(4) Let T be a quasi-ideal of S. For all $x \in S$, if $x \in TS \cap ST$, then $x \in T$. So $A(x) \leq \mu$. It follows that $A(x) \wedge \lambda \leq \mu \leq (A1_S)(x) \vee (1_S A)(x) \vee \mu$. If $x \notin TS \cap ST$, then $x \notin TS$ or $x \notin ST$. It can be obtained that $(A1_S)(x) \geq \lambda$ or $(1_S A)(x) \geq \lambda$. That is, $(A1_S)(x) \vee (1_S A)(x) \geq \lambda$. Hence $A(x) \wedge \lambda \leq \lambda \leq (A1_S)(x) \vee (1_S A)(x) \vee \mu$. It means that A is a (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S. ■

A (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup ($\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of S can be characterised by its cut set and open cut set.

Theorem 3.3 *Let A be an $\mathcal{N}$-fuzzy subset of S. Then A is a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup ($\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of S if and only if for all $t \in (\mu, \lambda]$, A_t is a subsemigroup ($\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of S whenever $A_t \neq \emptyset$.*

Proof. We only prove the cases of (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup and (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S.

(1) Let A be a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup of S. Then for all $t \in (\mu, \lambda]$ and $x, y \in A_t$, we have $A(xy) \wedge \lambda \leq A(x) \vee A(y) \vee \mu \leq t \vee \mu = t$. It implies that $A(xy) \leq t$ considering $\lambda > t$. So $xy \in A_t$. This indicates that A_t is a subsemigroup of S.
Conversely, let A_t be a subsemigroup of S for all $t \in (\mu, \lambda]$. If possible, let $A(x_0y_0) \wedge \lambda > A(x_0) \vee A(y_0) \vee \mu$ for some $x_0, y_0 \in S$. Put $t = A(x_0) \vee A(y_0) \vee \mu$, then $t \in (\mu, \lambda]$ and $x_0y_0 \notin A_t, x_0, y_0 \in A_t$. This is a contradiction. Hence for all $x, y \in S$, $A(xy) \wedge \lambda \leq A(x) \vee A(y) \vee \mu$. It means that A is a (λ, mu)-$\mathcal{N}$-fuzzy subsemigroup of S.
(2) Let A be a (λ, mu)-$\mathcal{N}$-fuzzy quasi-ideal of S and let $x \in AS \cap SA$, where $t \in (\mu, \lambda]$. Then there exist $a, b \in A_t$ and $s, t \in S$, such that $x = as = tb$. So we have

$$\begin{aligned} A(x) \wedge \lambda &\leq (A1_S)(x) \vee (1_SA)(x) \vee \mu \\ &\leq A(a) \vee 1_S(s) \vee 1_S(t) \vee A(b) \vee \mu \\ &= A(a) \vee A(b) \vee \mu \leq t \vee \mu \\ &= t < \lambda. \end{aligned}$$

and hence $A(x) \leq t$. So $x \in A$ and $AS \cap SA \subseteq A$. This indicates that A_t is a quasi-ideal of S.

Conversely, let A_t is a quasi-ideal of S for all $t \in (\mu, \lambda]$. If possible, let $A(x_0) \wedge \lambda > (A1_S)(x_0) \vee (1_SA)(x_0) \vee \mu$ for some $x_0 \in S$. Put $t = \frac{1}{2}[(A(x_0) \wedge \lambda) + ((A1_S)(x_0) \vee (1_SA)(x_0) \vee \mu)]$, then $t \in (\mu, \lambda], x_0 \notin A_t$ and $(A1_S)(x_0) \vee (1_SA)(x_0) < t$. So there exist $a, b, s, t \in S$, such that $x_0 = as = tb$ and $A(a) \vee A(b) < t$. It implies $x_0 \in AS \cap SA$. This is a contradiction. Hence for all $x \in S$, we have $A(x) \wedge \lambda \leq (A1_S)(x) \vee (1_SA)(x) \vee \mu$. It means that A is a (λ, μ)-$\mathcal{N}$-fuzzy quasi-ideal of S. This completes the proof. ∎

Similarly we have the following theorem.

Theorem 3.4 *Let A be an $\mathcal{N}$-fuzzy subset of S. Then A is a (λ, μ)-$\mathcal{N}$-fuzzy subsemigroup ($\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of S if and only if for all $t \in (\mu, \lambda]$, $A_{<t>}$ is a subsemigroup ($\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of S whenever $A_{<t>} \neq \emptyset$.*

References

[1] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems, vol. 5, no. 2 (1981), 203215.

[2] N. Kuroki, *On fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems, vol. 8, no. 1 (1982), 7179.

[3] N. Kuroki, *On fuzzy semigroups*, Information Sciences, vol. 53, no. 3 (1991), 203236.

[4] N. Kuroki, *Fuzzy generalized bi-ideals in semigroups*, Information Sciences, vol. 66, no. 3 (1992), 235 243.

[5] N. Kuroki, *On fuzzy semiprime quasi-ideals in semigroups*, Information Sciences, vol. 75, no. 3 (1993), 201211.

[6] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 1971, 512-517.

[7] L. A. Zadeh, *Fuzzy sets*, Information and Control, vol. 8 (1965), 338-353.

Received: September, 2011