Generalized $\mathcal{N}$-Fuzzy Ideals in Semigroups

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Abstract

In this paper, the concepts of $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal and $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal were introduced which can be regarded as a generalization of common correspondence concepts, and some properties of $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal and $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal were discussed.

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1 Introduction

The concept of fuzzy set, introduced by Zadeh [7], was applied to the theory of groups by Rosenfeld [6]. Since then, many scholars have studied the theorems of fuzzy subring and various fuzzy ideals. Kuroki studied fuzzy semigroup and fuzzy bi-ideal in a semigroup [1-5]. In this paper, the concepts of $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal and $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal were introduced which can be regarded as a generalization of common correspondence concepts, and some properties of $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal and $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal were discussed.

2 Preliminary Notes

A semigroup is more general algebraic system than the groups, and is widely applied in computer sciences. In the following statement, $S$ will stand for a semigroup and $\lambda$ and $\mu$ denote two constants such that $0 \leq \mu < \lambda \leq 1$.

By a subsemigroup $T$ of $S$ we mean a nonempty subset of $S$ such that $TT \subseteq T$. A nonempty subset $T$ of $S$ is called a left ideal (right ideal) of $S$ if $ST \subseteq T$ ($TS \subseteq T$). A nonempty subset $T$ of $S$ is called an ideal of $S$ if it is both a left ideal and a right ideal of $S$. A nonempty subset $T$ of $S$ is called a generalized bi-ideal of $S$ if $TST \subseteq T$. If the subset $T$ of $S$ is both a generalized bi-ideal and a subsemigroup of $S$, then $T$ is called a bi-ideal of $S$. 
A subsemigroup $T$ of $S$ is called an inner ideal if $STS \subseteq T$. A nonempty subset $Q$ of $S$ is called a quasi-ideal of $S$ if $QS \cap SQ \subseteq Q$.

Let $X$ be a nonempty set. By an $\mathcal{N}$-fuzzy subset $A$ of $X$ we mean a mapping from $X$ to the closed interval $[-1,0]$. If $A$ is an $\mathcal{N}$-fuzzy subset of $X$, then the cut set $A_t$ and open cut set $A_{<t}$ were defined as:

$$A_t := \{ x \in X \mid A(x) \leq t \}, \quad A_{<t} := \{ x \in X \mid A(x) < t \}.$$

**Definition 2.1** Let $A$ be an $\mathcal{N}$-fuzzy subset of $S$. Then $A$ is called a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup of $S$ if for all $x, y \in S$,

$$A(xy) \land \lambda \leq A(x) \lor A(y) \lor \mu.$$

**Definition 2.2** Let $A$ be an $\mathcal{N}$-fuzzy subset of $S$. Then $A$ is called a $(\lambda, \mu)$-generalized $\mathcal{N}$-fuzzy bi-ideal of $S$ if for all $x, y, z \in S$,

$$A(xyz) \land \lambda \leq A(x) \lor A(z) \lor \mu.$$

If $A$ is both a $(\lambda, \mu)$-generalized $\mathcal{N}$-fuzzy bi-ideal and an anti $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup of $S$, then $A$ is called a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal of $S$.

**Definition 2.3** Let $A$ be a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup of $S$. Then $A$ is called a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy inner ideal is for all $x, y, z \in S$,

$$A(xyz) \land \lambda \leq A(y) \lor \mu.$$

Let $A$ and $B$ two $\mathcal{N}$-fuzzy subsets of $S$. Then the $\mathcal{N}$-fuzzy subset $AB$ is defined as follows: if $x \in S$ can be expressed as $x = x_1x_2$, then $AB(x) := \inf \{ A(x_1) \lor B(x_2) \mid x_1, x_2 \in S \}$. Otherwise $AB(x) = 0$.

**Definition 2.4** Let $A$ be a $\mathcal{N}$-fuzzy subset of $S$. If for all $x \in S$, $A(x) \land \lambda \leq (A1_S)(x) \lor (1_SA)(x) \lor \mu$, then $A$ is called a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal of $S$, where $1_S : S \to \{-1,0\}$ is the characteristic $\mathcal{N}$-function of $S$ defined by

$$1_S(x) := \begin{cases} -1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

### 3 Main Results

**Theorem 3.1** Let $A$ be a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal of $S$. Then $A$ is both a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup and a $(\lambda, \mu)$-generalized $\mathcal{N}$-fuzzy bi-ideal of $S$ and hence $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal of $S$. 


Proof. For all $x, y, z \in S$, we have

$$A(xy) \land \lambda \leq (A_1S)(xy) \lor (1SA)(xy) \lor \mu$$
$$\leq A(x) \lor 1S(y) \lor 1S(x) \lor A(y) \lor \mu$$
$$= A(x) \lor A(y) \lor \mu.$$ 

$$A(xyz) \land \lambda \leq (A_1S)(xyz) \lor (1SA)(xyz) \lor \mu$$
$$\leq A(x) \lor 1S(yz) \lor 1S(xy) \lor A(z) \lor \mu$$
$$= A(x) \lor A(z) \lor \mu.$$ 

Hence $A$ is both a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup and a $(\lambda, \mu)$-generalized $\mathcal{N}$-fuzzy bi-ideal of $S$ and hence $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal bi-ideal of $S$. ■

**Theorem 3.2** Let $T$ be a nonempty subset of $S$ and let $A$ be an $\mathcal{N}$-fuzzy subset of $S$ such that

$$A(x) \begin{cases} \geq \lambda & \text{if } x \notin T, \\ \leq \mu & \text{if } x \in T, \end{cases}$$

for all $x \in S$. Then

1. $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup of $S$ if $T$ is a subsemigroup of $S$.
2. $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy bi-ideal of $S$ if $T$ is a bi-ideal of $S$.
3. $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy inner ideal of $S$ if $T$ is an inner ideal of $S$.
4. $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal of $S$ if $T$ is a quasi-ideal of $S$.

Proof. We only prove (1) and (4).

1. Let $T$ be a subsemigroup of $S$. For all $x, y \in S$, if $x, y \in T$, then $xy \in T$. So we have $A(xy) \leq \mu$ and $A(xy) \land \lambda \leq \mu = A(x) \lor A(y) \lor \mu$. If $x \notin T$ or $y \notin T$, then $A(x) \lor A(y) \lor \mu \geq \lambda$. It follows that $A(xy) \land \lambda \leq \lambda \leq A(x) \lor A(y) \lor \mu$. So $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup of $S$.

4. Let $T$ be a quasi-ideal of $S$. For all $x \in S$, if $x \in TS \cap ST$, then $x \in T$. So $A(x) \leq \mu$. It follows that $A(x) \land \lambda \leq \mu \leq (A_1S)(x) \lor (1SA)(x) \lor \mu$. If $x \notin TS \cap ST$, then $x \notin TS$ or $x \notin ST$. It can be obtained that $(A_1S)(x) \geq \lambda$ or $(1SA)(x) \geq \lambda$. That is, $(A_1S)(x) \lor (1SA)(x) \geq \lambda$. Hence $A(x) \land \lambda \leq \lambda \leq (A_1S)(x) \lor (1SA)(x) \lor \mu$. It means that $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal of $S$. ■

A $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup $(\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of $S$ can be characterised by its cut set and open cut set.

**Theorem 3.3** Let $A$ be an $\mathcal{N}$-fuzzy subset of $S$. Then $A$ is a $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup $(\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of $S$ if and only if for all $t \in (\mu, \lambda]$, $A_t$ is a subsemigroup $(\mathcal{N}$-fuzzy bi-ideal, $\mathcal{N}$-fuzzy inner ideal, $\mathcal{N}$-fuzzy quasi-ideal) of $S$ whenever $A_t \neq \emptyset$.

Proof. We only prove the cases of $(\lambda, \mu)$-$\mathcal{N}$-fuzzy subsemigroup and $(\lambda, \mu)$-$\mathcal{N}$-fuzzy quasi-ideal of $S$. 
(1) Let \( A \) be a \((\lambda, \mu)-N\)-fuzzy subsemigroup of \( S \). Then for all \( t \in (\mu, \lambda] \) and \( x, y \in A_t \), we have \( A(xy) \wedge \lambda \leq A(x) \vee A(y) \vee \mu \leq t \vee \mu = t \). It implies that \( A(xy) \leq t \) considering \( \lambda > t \). So \( xy \in A_t \). This indicates that \( A_t \) is a subsemigroup of \( S \).

Conversely, let \( A_t \) be a subsemigroup of \( S \) for all \( t \in (\mu, \lambda] \). If possible, let \( A(x_0y_0) \wedge \lambda > A(x_0) \vee A(y_0) \vee \mu \) for some \( x_0, y_0 \in S \). Put \( t = A(x_0) \vee A(y_0) \vee \mu \), then \( t \in (\mu, \lambda] \) and \( x_0y_0 \notin A_t \), \( x_0, y_0 \in A_t \). This is a contradiction. Hence for all \( x, y \in S \), \( A(xy) \wedge \lambda \leq A(x) \vee A(y) \vee \mu \). It means that \( A \) is a \((\lambda, \mu)-N\)-fuzzy subsemigroup of \( S \).

(2) Let \( A \) be a \((\lambda, mu)-N\)-fuzzy quasi-ideal of \( S \) and let \( x \in AS \cap SA \), where \( t \in (\mu, \lambda] \). Then there exist \( a, b \in A_t \) and \( s, t \in S \), such that \( x = as = tb \). So we have

\[
A(x) \wedge \lambda \leq (A1_S)(x) \vee (1_S A)(x) \vee \mu \\
\leq A(a) \vee 1_S(s) \vee 1_S(t) \vee A(b) \vee \mu \\
= A(a) \vee A(b) \vee \mu \leq t \vee \mu \\
= t < \lambda,
\]

and hence \( A(x) \leq t \). So \( x \in A \) and \( AS \cap SA \subseteq A \). This indicates that \( A_t \) is a quasi-ideal of \( S \).

Conversely, let \( A_t \) be a quasi-ideal of \( S \) for all \( t \in (\mu, \lambda] \). If possible, let \( A(x_0) \wedge \lambda > (A1_S)(x_0) \vee (1_S A)(x_0) \vee \mu \) for some \( x_0 \in S \). Put \( t = \frac{1}{2}[(A(x_0) \wedge \lambda) + ((A1_S)(x_0) \vee (1_S A)(x_0) \vee \mu)] \), then \( t \in (\mu, \lambda] \), \( x_0 \notin \bigcap A_t \) and \( (A1_S)(x_0) \vee (1_S A)(x_0) < t \). So there exist \( a, b, s, t \in S \), such that \( x_0 = as = tb \) and \( A(a) \vee A(b) < t \). It implies \( x_0 \in AS \cap SA \). This is a contradiction. Hence for all \( x \in S \), we have \( A(x) \wedge \lambda \leq (A1_S)(x) \vee (1_S A)(x) \vee \mu \). It means that \( A \) is a \((\lambda, \mu)-N\)-fuzzy quasi-ideal of \( S \). This completes the proof.

Similarly we have the following theorem.

**Theorem 3.4** Let \( A \) be an \( N \)-fuzzy subset of \( S \). Then \( A \) is a \((\lambda, \mu)-N\)-fuzzy subsemigroup (\( N \)-fuzzy bi-ideal, \( N \)-fuzzy inner ideal, \( N \)-fuzzy quasi-ideal) of \( S \) if and only if for all \( t \in (\mu, \lambda] \), \( A_{<t>} \) is a subsemigroup (\( N \)-fuzzy bi-ideal, \( N \)-fuzzy inner ideal, \( N \)-fuzzy quasi-ideal) of \( S \) whenever \( A_{<t>} \neq \emptyset \).

**References**


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