International Mathematical Forum, Vol. 7, 2012, no. 22, 1089 - 1098

Subtraction Algebras with Pseudo-Valuations

Kyung Ho Kim

Department of Mathematics, Chungju National University
Chungju 380-702, Korea
ghkim@cjnu.ac.kr

Yong Ho Yon

Innovation Center for Engineering Education, Mokwon University
Daejeon 302-729, Korea
yhyon@mokwon.ac.kr

Abstract

In this paper, Using Buşneag's model ([1, 2, 3]), we introduce the notion of pseudo-valuation (valuation) on a subtraction algebra and a pseudo-metric is induced by a pseudo-valuation on subtraction algebras.

Mathematics Subject Classification: 06F35, 03G25, 03C05

Keywords: (Weak) pseudo-valuation, valuation, pseudo-metric induced by pseudo-valuation

1 Introduction

Buşneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo-metric on a Hilbert algebra. Also, Buşneag [3] provided several theorems on extensions of pseudo-valuations. Buşneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extensions for these (using the model of Hilbert algebra ([3])). In this paper, Using Buşneag's model ([1, 2, 3]), we introduce the notion of pseudo-valuation (valuation) on a subtraction algebra and a pseudo-metric is induced by a pseudo-valuation on subtraction algebras.

2 Preliminaries

By a *subtraction algebra* we mean an algebra $(X;-)$ with a single binary operation "$-$" that satisfies the following identities:

(S1) $x-(y-x)=x$,

(S2) $x-(x-y)=y-(y-x)$,

(S3) $(x-y)-z=(x-z)-y$,

for any $x,y,z\in X$

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a\leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a\in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a\wedge b=a-(a-b)$; the relative complement b' of an element $b\in[0,a]$ is $a-b$; and if $b,c\in[0,a]$, then

$$\begin{aligned} b\vee c &= (b'\wedge c')'=a-((a-b)\wedge(a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

In a subtraction algebra, the following properties are true:

(p1) $(x-y)-y=x-y$,

(p2) $x-0=x$ and $0-x=0$,

(p3) $(x-y)-x=0$,

(p4) $x-(x-y)\leq y$,

(p5) $(x-y)-(y-x)=x-y$,

(p6) $x-(x-(x-y))=x-y$,

(p7) $(x-y)-(z-y)\leq x-z$,

(p8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$,

(p9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$,

(p10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$,

(p11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$, (p12) $(x - y) - z = (x - z) - (y - z)$,

for all $x, y, z \in X$.

A non-empty subset I of a subtraction algebra X is called a *subalgebra* if $x - y \in I$ for all $x, y \in I$.

A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,

(I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

3 Subtraction algebras with pseudo-valuations

In what follows, let X denote a subtraction algebra unless otherwise specified.

Definition 3.1. A real-valued function φ on X is called a *weak pseudo-valuation* on X if it satisfies for all $x, y \in X$, the following condition:

$$\varphi(x - y) \leq \varphi(x) + \varphi(y). \tag{1}$$

Example 3.2. Let $X = \{0, a, b, 1\}$ in which "$-$" is defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

It is easy to check that $(X; -)$ is a subtraction algebra. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix}.$$

Then φ is a weak pseudo-valuation on X.

Definition 3.3. A real-valued function φ on X is called a *pseudo-valuation* on X if it satisfies for all $x, y \in X$, the following condition:

$$\varphi(0) = 0 \tag{2}$$

$$\varphi(x) \leq \varphi(x - y) + \varphi(y). \tag{3}$$

A pseudo-valuation φ on X satisfying the following condition:

$$(\forall x \in X) \quad (x \neq 0 \Rightarrow \varphi(x) \neq 0) \tag{4}$$

is called a *valuation* on X.

Example 3.4. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b \\ 0 & 2 & 3 \end{pmatrix}$$

Then φ is a pseudo-valuation on X.

Example 3.5. Let $X = \{0, a, b, 1\}$ in which "$-$" is defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

It is easy to check that $(X; -)$ is a subtraction algebra. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Then φ is a pseudo-valuation on X.

Proposition 3.6. *For a pseudo-valuation φ on X, we have*

$$(\forall x \in X) \qquad (\varphi(x) \geq 0).$$

Proof. For any $x \in X$, we have $\varphi(0) = \varphi(0 - x) \geq \varphi(0) - \varphi(x)$, and so $\varphi(x) \geq 0$. $\square$

Proposition 3.7. *Let S be a subalgebra of X. For any positive real numbers t_1 and t_2 with $t_1 < t_2$, let φ_s be real-valued function on X defined by*

$$\varphi_s(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then φ_s is a weak pseudo-valuation on X.

Proof. Straightforward. □

Given an element a of a subtraction algebra X, the set

$$A(a) = \{x \in X \mid x \leq a\}$$

is called the *initial section* of X determined by a.

Corollary 3.8. *Let X be a subtraction algebra. For any $a \in X$, let φ be a real-valued function on X defined by*

$$\varphi_a(x) = \begin{cases} t_1 & \text{if } x \in A(a) \\ t_2 & \text{if } x \notin A(a) \end{cases}$$

for all $x \in X$ where t_1 and t_2 are real numbers with $t_2 > t_1 \geq 0$ and $A(a)$ is the initial section on X determined by a. Then φ_a is a weak pseudo-valuation on X.

Proposition 3.9. *For any pseudo-valuation φ on X, we have the following inequalities:*

(i) *φ is order preserving.*

(ii) $(\forall x, y \in X) \quad (\varphi(x - y) + \varphi(y - x) \geq 0.$

(iii) $(\forall x, y, z \in X) \quad \varphi(x - y) \leq \varphi(x - z) + \varphi(z - y).$

Proof. (i) Let $x \leq y$. Then $x - y = 0$. By (3.3), we get $\varphi(x) \leq \varphi(x-y)+\varphi(y) = \varphi(0) + \varphi(y)$, which implies $\varphi(x) \leq \varphi(y)$.

(ii) Let $x, y \in X$. By (3.3), $\varphi(x - y) \geq \varphi((x) - \varphi(y)$ and $\varphi(y - x) \geq \varphi((y) - \varphi(x)$. It follows that $\varphi(x - y) + \varphi(y - x) \geq 0$.

(iii) Let $x, y, z \in X$. Note that $(x - y) - (z - y) \leq x - z$. By (i) and (3.3), we have

$$\varphi(x - z) \geq \varphi((x - y) - (z - y)) \geq \varphi(x - y) - \varphi(z - y).$$

Hence $\varphi(x - y) \leq \varphi(z - y) + \varphi(x - z) = \varphi(x - z) + \varphi(z - y)$. □

Proposition 3.10. *For any pseudo-valuation φ on X, we have the following inequalities:*

(i) $(\forall x, y \in X) \quad (\varphi(x - y) \leq \varphi(x - y) + \varphi(y - x)).$

(ii) $(\forall x, y, z \in X) \quad (\varphi(x - y) \leq \varphi(x - y) + \varphi(y)).$

Proof. (i) By (3.3) and (p5), $\varphi(x-y) \le \varphi((x-y)-(y-x)) + \varphi(y-x) = \varphi(x-y) + \varphi(y-x)$.

(ii) By (3.3) and (p1), $\varphi(x-y) \le \varphi((x-y)-y) + \varphi(y) = \varphi(x-y) + \varphi(y)$. □

Proposition 3.11. *Every pseudo-valuation φ on X satisfies the following implication:*

$$(\forall x, y, z \in X) \quad ((x-y)-z = 0 \Rightarrow \varphi(x) \le \varphi(y) + \varphi(z)). \tag{5}$$

Proof. Let $x, y, z \in X$ be such that $(x-y)-z = 0$. It follows from (3.2) and (3.3) that

$$\varphi(x-y) \le \varphi((x-y)-z) + \varphi(z) = \varphi(0) + \varphi(z) = \varphi(z)$$

so that

$$\varphi(x) \le \varphi(x-y) + \varphi(y) \le \varphi(y) + \varphi(z).$$

□

The following corollary can be proved by induction

Corollary 3.12. *Let φ be a pseudo-valuation on X. If*

$$(...((x-a_1)-a_2)-...)-a_n = 0,$$

we get $\varphi(x) \le \sum_{k=1}^{n} \varphi(a_k)$.

Proposition 3.13. *Every real-valued function φ on X satisfying (3.2) and (3.5) is a pseudo-valuation on X.*

Proof. By (p4), we have $(x-(x-y))-y = 0$ for all $x, y \in X$. Hence, it follows from (3.5) that $\varphi(x) \le \varphi(x-y) + \varphi(y)$. Therefore φ is a pseudo-valuation on X. This completes the proof. □

Proposition 3.14. *In a subtraction algebra X, every pseudo-valuation is a weak pseudo-valuation.*

Proof. Let φ be a pseudo-valuation on X. Since $((x-y)-x)-y = ((x-x)-y)-y = (0-y)-y = 0$ for all $x, y \in X$, we have

$$\begin{aligned} 0 =& \varphi(0) = \varphi(((x-y)-x)-y) \\ &\ge \varphi((x-y)-x) - \varphi(y) \\ &\ge \varphi(x-y) - \varphi(x) - \varphi(y). \end{aligned}$$

Hence $\varphi(x-y) \le \varphi(x) + \varphi(y)$, and so φ is a weak pseudo-valuation on X. This completes the proof. □

The following example shows that the converse of Theorem 3.14 may not be true.

Example 3.15. The weak pseudo-valuation φ in Example 3.2 is not a pseudo-valuation on X since $\varphi(a) = 3 \nleq 2 = \varphi(a-1) + \varphi(1)$.

Theorem 3.16. *If a real-valued function φ on X satisfies the condition (3.2) and*

$$(\forall x, y, z \in X) \quad (\varphi(((x-y)-y)-z) \geq \varphi(x-y) - \varphi(z)), \tag{6}$$

φ is a pseudo-valuation on X.

Proof. Taking $y = 0$ in (6), we have

$$\varphi(x-z) = \varphi(((x-0)-0)-z) \geq \varphi(x-0) - \varphi(z) = \varphi(x) - \varphi(z).$$

Hence φ is a pseudo-valuation on X. □

Theorem 3.17. *Let a real-valued function φ be a pseudo-valuation on X. Then the set*

$$I = \{x \in X \mid \varphi(x) = 0\}$$

is an ideal of X.

Proof. Obviously, $0 \in I$. Let $x, y \in X$ be such that $x - y \in I$ and $y \in I$. Then $\varphi(x-y) = 0$ and $\varphi(y) = 0$. It follows from (3.5) that $\varphi(x) \leq \varphi(x-y)+\varphi(y) = 0$ so that $\varphi(x) = 0$. Hence $x \in I$, which shows that I is an ideal of X. □

By a *pseudo-metric* we mean a positive function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(i) $d(x, x) = 0$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$. for all $x, y \in X$.

For a real-valued function φ on X, define a mapping $d_\varphi : X \times X \to \mathbb{R}$ by

$$d_\varphi(x, y) = \varphi(x-y) + \varphi(y-x)$$

for all $(x, y) \in X \times X$.

Theorem 3.18. *Let X be a subtraction algebra. If a real-valued function φ on X is a pseudo-valuation on X, d_φ is a pseudo-metric on X, and so (X, d_φ) is a pseudo-metric space.*

Proof. Obviously, $d_\varphi(x,y) \geq 0, d_\varphi(x,x) = 0$ and $d_\varphi(x,y) = d_\varphi(y,x)$ for all $x, y \in X$. Using Proposition 3.9, we have

$$\begin{aligned} d_\varphi(x,y) + d_\varphi(y,z) &= [\varphi(x-y) + \varphi(y-x)] + [\varphi(y-z) + \varphi(z-y)] \\ &= [\varphi(x-y) + \varphi(y-z)] + [\varphi(z-y) + \varphi(y-x)] \\ &\geq \varphi(x-z) + \varphi(z-x) = \varphi(x-z). \end{aligned}$$

Therefore, (X, d_φ) is a pseudo-metric space. □

Proposition 3.19. *Let X be a subtraction algebra. Then every pseudo-metric d_φ induced by pseudo-valuation φ satisfies the following inequality.*

(i) $d_\varphi(x,y) \geq d_\varphi(x-a, y-a)$,

(ii) $d_\varphi(x,y) \geq d_\varphi(a-x, a-y)$,

(iii) $d_\varphi(x-y, a-b) \leq d_\varphi(x-y, a-y) + d_\varphi(a-y, a-b)$, *for all* $x, y, a, b \in X$.

Proof. (i) Let $x, y, a \in X$. Since $((y-a)-(x-a))-(y-x) = 0$ and $((x-a)-(y-a))-(x-y) = 0$ from (p7), it follows from Proposition 3.9(i) that $\varphi(y-x) \geq \varphi((y-a)-(x-a))$ and $\varphi(x-y) \geq \varphi((x-a)-(y-a))$ and so that

$$\begin{aligned} d_\varphi(x,y) &= \varphi(x-y) + \varphi(y-x) \\ &\geq \varphi((x-a)-(y-a)) + \varphi((y-a)) - (x-a)) \\ &= d_\varphi(x-a, y-a). \end{aligned}$$

(ii) It is similar to the proof of (i).

(iii) Using Proposition 3.9(iii), we have

$$\varphi((x-y)-(a-b)) \leq \varphi((x-y)-(a-y)) + \varphi((a-y)+(a-b))$$

and

$$\varphi((a-b)-(x-y)) \leq \varphi((a-b)-(a-y)) + \varphi((a-y)-(x-y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_\varphi(x-y, a-b) &= \varphi((x-y)-(a-b)) + \varphi((a-b)-(x-y)) \\ &\leq [\varphi((x-y)-(a-y)) + \varphi((a-y)) - (a-b))] \\ &\quad + [\varphi((a-b)-(a-y)) + \varphi((a-y)) - (x-y))] \\ &= [\varphi((x-y)-(a-y)) + \varphi((a-y)) - (x-y))] \\ &\quad + [\varphi((a-b)-(a-y)) + \varphi((a-y)) - (a-b))] \\ &= d_\varphi(x-y, a-y) + d_\varphi(a-y, a-b) \end{aligned}$$

for all $x, y, a, b \in X$. □

Theorem 3.20. *For a real-valued function φ on X, if d_φ is a pseudo-metric on X, we have $(X \times X, d^*_\varphi)$ is a pseudo-metric space, where*

$$d^*_\varphi((x,y),(a,b)) = max\{d_\varphi(x,a), d_\varphi(y,b)\} \tag{7}$$

for all $(x,y),(a,b) \in X \times X$.

Proof. Suppose d_φ is a pseudo-metric on X. For any $(x,y),(a,b) \in X \times X$, we have

$$d^*_\varphi((x,y),(x,y)) = \max\{d_\varphi(x,x), d_\varphi(y,y)\} = 0$$

and

$$\begin{aligned} d^*_\varphi((x,y),(a,b)) &= \max\{d_\varphi(x,a), d_\varphi(y,b)\} \\ &= \max\{d_\varphi(a,x), d_\varphi(b,y)\} \\ &= d^*_\varphi((a,b),(x,y)). \end{aligned}$$

Now let $(x,y),(a,b),(u,v) \in X \times X$. Then

$$\begin{aligned} d^*_\varphi((x,y),(u,v)) + d^*_\varphi((u,v),(a,b)) &= \max\{d_\varphi(x,u), d_\varphi(y,v)\} + \max\{d_\varphi(u,a), d_\varphi(v,b)\} \\ &\geq \max\{d_\varphi(x,u) + d_\varphi(u,a), d_\varphi(y,v) + d_\varphi(v,b)\} \\ &\geq \max\{d_\varphi(x,a), d_\varphi(y,b)\} \\ &= d^*_\varphi((x,y),(a,b)). \end{aligned}$$

Therefore $(X \times X, d^*_\varphi)$ is a pseudo-metric space. □

Corollary 3.21. *If $\varphi : X \to \mathbb{R}$ is a pseudo-valuation on X, then $(X \times X, d^*_\varphi)$ is a pseudo-metric space.*

Theorem 3.22. *Let X be a subtraction algebra. If $\varphi : X \to \mathbb{R}$ is a valuation on X, (X, d_φ) is a metric space.*

Proof. Suppose that φ is a valuation on X. Then (X, d_φ) is a metric space by Theorem 3.18. Let $x, y \in X$ be such that $d_\varphi(x,y) = 0$ and $d_\varphi(y,x) = 0$. Then $0 = d_\varphi(x,y) = d_\varphi(x-y)+d_\varphi(y-x)$, and so $\varphi(x-y) = 0$ and $\varphi(y-x) = 0$. Now since φ is a valuation on X, it follows that $x - y = 0$ and $y - x = 0$. Therefore we have $x = x - (y - x) = x - 0 = x - (x - y) = y - (y - x) = y - 0 = y$. This implies that (X, d_φ) is a metric space. □

Theorem 3.23. *Let X be a subtraction algebra. If $\varphi : X \to \mathbb{R}$ is a valuation on X, $(X \times X, d^*_\varphi)$ is a metric space.*

Proof. Note from Corollary 3.20 that $(X \times X, d^*_\varphi)$ is a pseudo-metric space. Let $(x,y),(a,b) \in X \times X$ be such that $d^*_\varphi((x,y),(a,b)) = 0$. Then

$$0 = d^*_\varphi((x,y),(a,b)) = \max\{d^*_\varphi(x,a), d^*_\varphi(y,b))\},$$

and so $d_\varphi(x,a) = 0 = d_\varphi(y,b)$ since $d_\varphi(x,y) \geq 0$ for all $(x,y) \in X \times X$. Hence

$$0 = d_\varphi(x,a) = \varphi(x-a) + \varphi(a-x)$$

and

$$0 = d_\varphi(y,b) = \varphi(y-b) + \varphi(b-y).$$

Hence it follows that $= \varphi(x-a) == \varphi(a-x)$ and $= \varphi(y-b) = 0 == \varphi(b-y)$ so that $x-a = 0 = a-x$ and $y-b = 0 = b-y$, which implies $x = a$ and $y = b$. Hence we get $(x,y) = (a,b)$ so that $(X \times X, d^*_\varphi)$ is a metric space. □

Theorem 3.24. *Let X be a subtraction algebra. If $\varphi : X \to \mathbb{R}$ is a valuation on X, the operation " $-$ " is uniformly continuous.*

Proof. For any $\epsilon > 0$, if $d^*_\varphi((x,y),(a,b)) < \frac{\epsilon}{2}$, we have $d_\varphi(x,a) < \frac{\epsilon}{2}$ and $d_\varphi(y,b) < \frac{\epsilon}{2}$. Using Proposition 3.19, we get

$$\begin{aligned} d_\varphi(x-y),(a-b) &\leq d_\varphi(x-y, a-y) + d_\varphi(a-y, a-b) \\ &\leq d_\varphi((x,a) + d_\varphi(y,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, the operation $- : X \times X \to X$ is uniformly continuous. □

Acknowledgement

The research was supported by a grant from the Academic Research Program of Chungju National University in 2012.

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[2] C. Buşneag, *Valuations on residuated lattices*, An. Univ. Craiova Ser. Math. Inform. **34** (2007), 21-28

[3] C. Buşneag, *Hilbert algebra with valuations* , Math. Japon. **44** (1996), 285-289

[4] C. Buşneag, *On extensions of pseudo-valuations on Hilbert algebra*, Discrete Math. **263** (2003), 11-24

[5] B. M. Schein, *Difference Semigroups,* Comm. in Algebra, **20** (1992), 2153–2169.

[6] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

Received: October, 2011