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Solution for Time-Fractional Coupled Klein-Gordon Schrodinger Equation Using Decomposition Method

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Abstract

The time-fractional coupled Klein-Gordon Schrodinger equation is obtained from the coupled Klein-Gordon Schrodinger equation by replacing the order time derivative with a fractional derivative of order $\alpha \in (1,2]$, $\beta \in (0,1]$. The fractional derivative is described in Caputo sense. In this study a system of time-fractional coupled Klein-Gordon Schrodinger equation is considered with the initial values and the solution are presented using Adomian decomposition method. The solutions of the equation are presented and the figures show the effectiveness and good accuracy of the proposed method.

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1. Introduction

Since Adomian firstly proposed the decomposition method[1] at the beginning of 1980s, the algorithm has been widely and effectively used for solving the analytic solutions of physically significant equations arranging from linear to nonlinear, from ordinary differential to partial differential, from integer to fractional, etc [1,4,6,7,20,24,25,28]. Using this method, there is no need to take any special technique and the realistic solution can be easily obtained in the form of a rapidly convergent infinite series with each term being computed conveniently.

Fractional calculus is very old concept dating back to 17th century which involves fractional integration and differentiation. In the last few decades, fractional calculus has found many applications in various fields of physical sciences such as viscoelasticity, diffusion, control, relaxtion processor and so on [5,8,9,10,11,13,15,26]. The modified decomposition method (MDM) is used here for obtaining the numerical solutions of the time-fractional coupled nonlinear Klein-Gordon Schrodinger equation.

In this paper, the time-fractional coupled nonlinear Klein-Gordon-Schrodinger equation is considered in the following form:

$$\begin{cases} D_t^{\alpha} u - c^2 u_{xx} + u + |v|^2 = 0, \\ i D_t^{\beta} v + v_{xx} + uv = 0. \end{cases} \tag{1.1}$$

Here α, β are the parameters standing for the order of the fractional derivatives which satisfy $m-1 < \alpha \le m$, $n-1 < \beta \le n$ where $m = 2, n = 1$, and $t > 0$. When $\alpha = 2$ and $\beta = 1$, the fractional equation reduces to the classical coupled Klein-Gordon Schrodinger equation.

The paper is organized as follow; In Section 2, some necessary details on the fractional calculus are provided. In Section 3, the coupled Klein-Gordon equation with time-fractional derivative is studied with ADM. In Section 4 and 5, illustrative example is given and figures are used to show the efficiency as well as the accuracy of approximate results achieved. Finally, conclusions are presented in Section 6.

2. Description of the Fractional Calculus

The mathematical definition of fractional calculus has been the subject of several different approaches [14,15,16]. The most frequently encountered definition of an integral of fractional order is the Riemann-Liouville integral, in which the fractional order integral is defined as:

$$D_t^{-q} f(t) = \frac{d^{-q} f(t)}{dt^{-q}} = \frac{1}{\Gamma(q)} \int_0^t \frac{f(x)}{(t-x)^{1-q}} dx, \tag{2.1}$$

while the definition of fractional order derivative is;

$$D_t^q f(t) = \frac{d^n}{dt^n}\left(\frac{d^{-(n-q)} f(t)}{dt^{-(n-q)}}\right) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(x)}{(t-x)^{1-n+q}} dx, \tag{2.2}$$

where q ($q > 0$ and $q \in R$) is the order of the operation and n is an integer that satisfies $n-1 < q \le n.$

The fractional derivative of $f(t)$ in the caputo sense is defined by:

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, \tag{2.3}$$

where α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation and n is an integer that satisfies $n-1 < q < n.$

3. The Time-fractional Coupled Klein-Gordon Schrodinger Equation and its Solution

In this paper, the coupled fractional K-G-S equation (1.1) was considered in the operator form:

$$\begin{cases} D_t^\alpha u = c^2 L_{xx} u - u - N(u,v), \\ D_t^\beta v = i L_{xx} v + i M(u,v), \end{cases} \tag{3.1}$$

where $D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$, $D_t^\beta = \dfrac{\partial^\beta}{\partial t^\beta}$ are the Caputo derivative of order α and β, respectively, and $L_{xx} = \dfrac{d^2}{dx^2}$ symbolize the linear differential operator and the notations $N(u,v) = |v|^2$ and $M(u,v) = uv$ symbolize the nonlinear operator.

Furthermore, the following initial conditions have been considered:

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad v(x,0) = h(x).$$

where $f(x), g(x), h(x)$ are given functions.

Now Eq. (3.1) can be written as:

$$\begin{cases} L_{tt}u = D_t^{(m-\alpha)}(c^2 L_{xx}u - u - N(u,v)) & m-1 < \alpha \le m, \\ L_t v = D_t^{(n-\beta)}(iL_{xx}v + iM(u,v)) & n-1 < \beta \le n \quad . \end{cases} \tag{3.2}$$

where $L_{tt} = \dfrac{\partial^m}{\partial t^m}$, $L_t = \dfrac{\partial^n}{\partial t^n}$ which are easily invertible linear operators, and $D_t^{(m-\alpha)}(.)$, $D_t^{(n-\beta)}(.)$ are the Riemann-Liouville derivative of order $(m-\alpha)$ and $(n-\beta)$, respectively.

The Adomian decomposition method [2,3,4] assumes an infinite series solutions for unknown function $u(x,t)$ and $v(x,t)$ giving by;

$$\begin{cases} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \end{cases} \tag{3.3}$$

Therefore, using the Adomian decomposition method and applying the two-fold integration inverse operator L_{tt}^{-1}, L_t^{-1} to the system (3.2) and using the specified initial conditions, the following can be written:

$$\begin{cases} u(x,t) = u(x,0) + tu_t(x,0) + L_{tt}^{-1}(D_t^{(m-\alpha)}(c^2 L_{xx}u - u - N(u,v))) \\ v(x,t) = v(x,0) + L_t^{-1}(D_t^{(n-\beta)}(iL_{xx}v + M(u,v))) \end{cases}$$

where $L_{tt}(u(x,0) + tu_t(x,0)) = 0.and \quad L_t(v(x,0)) = 0.$ Nonlinear operators $N(u,v) = |v|^2$ and $M(u,v) = uv$ by the infinite series of Adomian polynomials.

By insetting the initial conditions in (3.4) the zeroth component u_0 and v_0, can be identified, then the subsequent component can be obtained by the following recursive equations:

$$\begin{cases} u_{n+1} = L_{tt}^{-1}(D_t^{m-\alpha}(c^2 L_{xx}u_n - u_n - A_n)), \\ v_{n+1} = L_t^{-1}(D_t^{n-\beta}(iL_{xx}v_n + iB_n)). \end{cases} \tag{3.4}$$

Recently, Wazwaz[29] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [21-25], he assumed that if the zeroth component $u_0 = h$ and the function h could be divided into two parts such as h_1 and h_2, one can formulate the recursive algorithm for u_0 and general term in the form of the modified recursive scheme as follows:

$$\begin{aligned} u_0 &= h_1, \\ u_1 &= h_2 + L_{tt}^{-1}(D_t^{m-\alpha}(c^2 L_{xx} u_0 - u_0 - A_0)), \\ u_{n+1} &= L_{tt}^{-1}(D_t^{m-\alpha}(c^2 L_{xx} u_n - u_n - A_n)), \quad n \geq 1. \end{aligned} \tag{3.5}$$

Similarly:

$$\begin{aligned} v_0 &= g_1, \\ v_1 &= g_2 + L_t^{-1}(D_t^{n-\beta}(c^2 L_{xx} v_0 - v_0 - B_0)), \\ v_{n+1} &= L_t^{-1}(D_t^{n-\beta}(c^2 L_{xx} v_n - v_n - B_n)), \quad n \geq 1. \end{aligned} \tag{3.6}$$

The decomposition series (3.3) solutions generally converge very rapidly in the real physical problem [4]. The practical solutions are the $n-$term approximations ϕ_n and φ_n :

$$\phi_n = \sum_{i=0}^{n-1} u_i(x,t), \qquad \varphi_n = \sum_{i=0}^{n-1} v_i(x,t), \qquad n \geq 1. \tag{3.7}$$

with

$$\lim_{n\to\infty} \phi_n = u(x,t), \qquad \lim_{n\to\infty} \varphi_n = v(x,t).$$

4. Implementing the Method

First the application of time-fractional coupled K-G-S equations (3.2) was considered with the initial conditions:

$$\begin{gathered} u(x,t) = -14p^2 - 6p^2 \tanh^2(px), \qquad u_t(x,t) = 24kp^3 \tanh(px)\operatorname{sec}h^2(px), \\ v(x,0) = (-\frac{7}{2}p + 6p\tanh^2(px))e^{ikx}, \end{gathered} \tag{4.1}$$

Where p and k are arbitrary constants. When $\alpha = 2, \beta = 1$ the exact solution is:

$$u(x,t) = -14p^2 \tanh^2(px) + 24kp^3 t \tanh(px)\sec h^2(px) + p^2(-1+4k^2p^2)t^2$$
$$\sec h^5(px)(-9\cosh(px) + 3\cosh(3px) + 4kpt(-11\sinh(px) + \sinh(3px)))$$
$$- p^2t^2(-7 + 4kpt \sec h^2(px)\tanh(px) - 3\tanh^2(px)) + ...$$
$$v(x,t) = (-\frac{7}{2}p + 6p\tanh^2(px))e^{ikx} + \frac{1}{2}ie^{ikx}pt(-5k^2 + 12\sec h^2(px)(k^2 - 4p^2$$
$$+ 6p^2 \sec h^2(px) + 4ikp\tanh(px))) + ...$$

Using (3.8) and (3.9) with (3.5) and (3.6) respectively and considering $c^2 = \frac{4k^2p^2 - 1}{p^2}$ for the time-fractional coupled K-G-S equation and by initial conditions (4.1) result in:

$$u_0 = 0,$$

$$u_1 = u(x,0) + tu_t(x,0) + L_{tt}^{-1}(D_t^{m-\alpha}(c^2 L_{xx}u_0 - u_0 - A_0))$$
$$= -14p^2 - 6p^2\tanh^2(px) + 24tkp^3\tanh(px)\sec h^2(px),$$

$$u_2 = L_{tt}^{-1}(D_t^{m-\alpha}(c^2 L_{xx}u_1 - u_1 - A_1))$$
$$= \frac{1}{p^2}((4k^2p^2 - 1)(-12p^4(1-\tanh^2(px))^2 \frac{t^\alpha}{\Gamma(\alpha+1)} + 24p^4\tanh^2(px)(1-\tanh^2(px))t^2$$
$$\frac{t^\alpha}{\Gamma(\alpha+1)} - 192kp^5\tanh(px)(1-\tanh^2(px))\sec h^2(px)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 96kp^5\tanh^3(px)\sec h^2(px)$$
$$\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 14p^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 6p^2\tanh^2(px)\frac{t^\alpha}{\Gamma(\alpha+1)} - 24kp^3\tanh(px)\sec h^2(px)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$v_0 = 0,$$

$$v_1 = v(x,0) + L_t^{-1}(D_t^{n-\beta}(iL_{xx}v_0 + iB_0)) = (-\frac{7}{2}p + 6\tanh^2(px))e^{ikx},$$

$$v_2 = L_t^{-1}(D_t^{n-\beta}(iL_{xx}v_1 + iB_1))$$
$$= 12p^3(1-\tanh^2(px))^2 e^{ikx} - 24p^3\tanh^2(px)(1-\tanh^2(px))e^{ikx} + 24p^2$$
$$\tanh(px)(1-\tanh^2(px))e^{ikx}ik + (-\frac{7}{2}p - 6p\tanh^2(px))e^{ikx}k^2)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

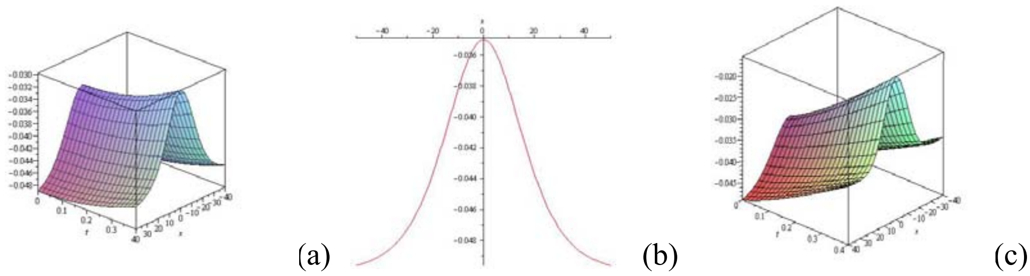
And so on. In this manner the other components of the decomposition series can be easily obtained of which $u(x,t)$ and $v(x,t)$ will evaluated in a series form:

$$\begin{aligned}
u(x,t) &= -14p^2 - 6p^2\tanh^2(px) + 24tkp^3\tanh(px)\sec h^2(px) \\
&+ \frac{1}{p^2}((4k^2p^2-1)(-12p^4(1-\tanh^2(px))^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 24p^4\tanh^2(px) \\
&(1-\tanh^2(px))t^2\frac{t^\alpha}{\Gamma(\alpha+1)} - 192kp^5\tanh(px)(1-\tanh^2(px))\sec h^2(px) \\
&\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 96kp^5\tanh^3(px)\sec h^2(px)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 14p^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 6p^2 \\
&\tanh^2(px)\frac{t^\alpha}{\Gamma(\alpha+1)} - 24kp^3\tanh(px)\sec h^2(px)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
v(x,t) &= (-\frac{7}{2}p + 6\tanh^2(px))e^{ikx} \\
&+ 12p^3(1-\tanh^2(px))^2e^{ikx} - 24p^3\tanh^2(px)(1-\tanh^2(px))e^{ikx} + 24p^2 \\
&\tanh(px)(1-\tanh^2(px))e^{ikx}ik + (-\frac{7}{2}p - 6p\tanh^2(px))e^{ikx}k^2)\frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \tag{4.3}$$

5. Numerical Results and Discussions

In the present numerical experiments (4.2) and (4.3) have been used to draw the graphs as shown in Figs 1 and 2. In the present numerical computation we have assumed $p = 0.05$ and $k = 0.05$.



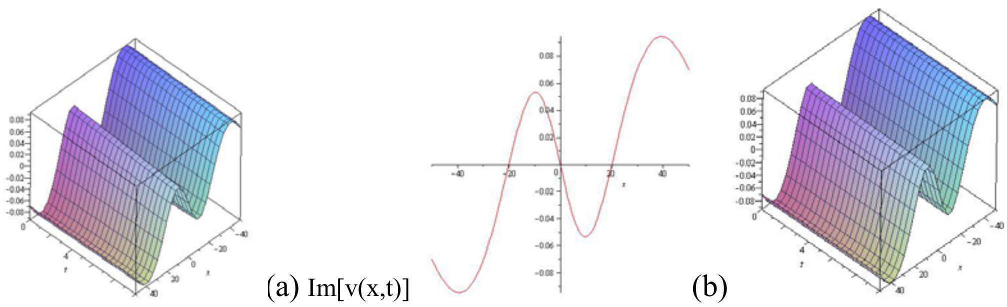
(a) (b) (c)

Figure 1: (a) The decomposition method solution for $u(x,t)$ when $\alpha = 2$, (b) corresponding solution for $u(x,t)$ when $t = 0$, (c) The decomposition method solution for $u(x,t)$ when $\alpha = 1.2$.

t	x	$\alpha=1.2$	$\alpha=1.7$	$\alpha=2$	exact
0.05	5	-0.03442664	-0.03566337	-0.03582426	-0.03582426
0.1	10	-0.03552396	-0.03759473	-0.03796438	-0.03796438
0.15	15	-0.03756830	-0.04008291	-0.04062312	-0.04062312
0.2	20	-0.03916814	-0.04224810	-0.04300326	-0.04300326
0.25	25	-0.03958756	-0.04357456	-0.04465856	-0.04465856
0.3	30	-0.03885489	-0.04398955	-0.04550997	-0.04550997
0.35	35	-0.03731233	-0.04365577	-0.04567652	-0.04567652
0.4	40	-0.03529167	-0.04278063	-0.04532456	-0.04532456

Table1: Approximate solution of (3.1) for some values of α using the 3-term ADM, respectively when $\alpha=1.2,1.7$ and $\alpha=2$.

The table 1 show that when α close to 2, approximate solutions of u are convergent to the exact solution.



(a) Im[v(x,t)] (b) (c)

Figure 2: (a) The decomposition method solution for Im[$v(x,t)$] when $\beta=1$, (b) corresponding solution for Im[$v(x,t)$] when $t=0$,(c) The decomposition method solution for Im[$v(x,t)$].when $\beta=0.3$.

t	x	$\beta = 0.1$	$\beta = 0.7$	$\beta = 1$	exact
0.05	**5**	-0.038023678	-0.03870127	-0.03879090	-0.03879090
0.1	**10**	-0.05210514	-0.05290081	-0.05305544	-0.05305544
0.15	**15**	-0.03610027	-0.03655994	-0.03667263	-0.03667263
0.2	**20**	-0.00072044	-0.00078925	-0.00080934	-0.00080934
0.25	**25**	0.03848417	0.03865076	0.03870663	0.03870663
0.3	**30**	0.07011118	0.07035672	0.07044988	0.07044988
0.35	**35**	0.08887197	0.08910614	0.08920559	0.08920559
0.4	**40**	0.09391626	0.09410352	0.09419207	0.09419207

Table2: Approximate solution of (1) for some values of β using the 3-term ADM, respectively when $\beta = 0.1, 0.7$ and $\beta = 1$.

The table 2 show that when β close to 1, approximate solutions of v are convergent to the exact solution.

6. Conclusions

In this paper, combining the Caputo fractional derivative, the ADM has been successfully extended to derive the explicit numerical solutions for the time-fractional coupled Klein-Gordon Schrodinger equation with initial condition, the above procedure show that:

1) The ADM is an efficient and powerful method in solving a wide class of equations, in particular, coupled fractional order equations.
2) The method is straightforward without any restrictive assumptions and special techniques.
3) The continuity of the solution depend on the time-fractional derivative operator and the convergent speed is related with only two terms.

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