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Minimal Quasi-Ideals in Ordered Γ-Semigroups

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Abstract

In this note, we give two characterizations of minimal quasi-ideals of ordered Γ-semigroups. The results generalize the results presented in [5].

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1 Preliminaries

It is known that *po*-Γ-semigroup is a generalization of Γ-semigroup and Γ-semigroup is a generalization of semigroup. Let S and Γ be two nonempty sets. Then S is called a Γ-*semigroup* if there is a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and all $\gamma, \beta \in \Gamma$.

Let S be a semigroup and Γ a nonempty set. For $x, y \in S$ and $\gamma \in \Gamma$, define $x\gamma y = xy$. Then S is a Γ-semigroup.

Let S be a Γ-semigroup. For $A, B \subseteq S$, let

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

For $x \in S$, let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A$.

Let S be a Γ-semigroup. A nonempty subset A of S is called a *left (resp. right) ideal* of S if $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). If A is both left and right ideal of S, then A is called an *ideal* of S. A nonempty subset Q of S is called a *quasi-ideal* of S if $Q\Gamma S\cap S\Gamma Q\subseteq Q$. Every right ideal and left ideal is a Quasi-ideal. A quasi-ideal Q of a Γ-semigroup S is called a *minimal quasi-ideal* of S if Q dose not properly contain any quasi-ideal of S.

The Green's relations $\mathcal{L},\mathcal{R},\mathcal{H}$ on a Γ-semigroup S are defined by (i) $a\mathcal{L}b$ if $S^1\Gamma a=S^1\Gamma b$ where $S^1\Gamma a=S\Gamma a\cup\{a\}$. (ii) $a\mathcal{R}b$ if $a\Gamma S^1=b\Gamma S^1$ where $a\Gamma S^1=a\Gamma S\cup\{a\}$. (iii) $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$.

In 2009, the authors [5] gave some characterizations of minimal quasi-ideal of a Γ-semigroup. Namely,

Theorem 1.1 *[5, Theorem 3.1] Let Q be a quasi-ideal of a Γ-semigroup S. Then Q is a minimal quasi-ideal of S if and only if Q is the intersection of a minimal left ideal L and a minimal right ideal R of S.*

Theorem 1.2 *[5, Theorem 3.5] Let Q be a quasi-ideal of a Γ-semigroup S. Then*

Q is a minimal quasi-ideal of S if and only if Q is a $\mathcal{H}$-class.

The purpose of this note is to generalize Theorem 1.1 and Theorem 1.2 using the concept of *po*-Γ-semigroup introduced by Sen and Saha [6].

Definition 1.3 *[1] A Γ-semigroup S is called an ordered Γ-semigroup (po-Γ-semigroup) if there is a relation $\leq$ on S such that $x\leq y$ implies $x\gamma z\leq y\gamma z$ and $z\gamma x\leq z\gamma y$ for any $x,y,z\in S$ and all $\gamma\in\Gamma$.*

Let S be a *po*-Γ-semigroup. For $A\subseteq S$, let

$$(A]=\{x\in S\mid x\leq a \text{ for some } a\in A\}.$$

See [3], for all $A,B\subseteq S$, we have

1) $A\subseteq(A]$,
2) If $A\subseteq B$, then $(A]\subseteq(B]$,
3) $(A]\Gamma(B]\subseteq(A\Gamma B]$,
4) $(A\cup B]=(A]\cup(B]$,
5) $(A\cap B]\subseteq(A]\cap(B]$.

Definition 1.4 *[1] Let S be a po-Γ-semigroup. A nonempty subset A of S is called a left (resp. right) ideal of S if*

(i) *$S\Gamma A\subseteq A$ (resp. $A\Gamma S\subseteq A$).*

(ii) *If $x\in A$ and $y\in S$ such that $y\leq x$, then $y\in A$.*

If A is both left and right ideal of S, then A is called an ideal of S.

It is easy to check that for any nonempty subset A of S, $(S\Gamma A]$ is a left ideal of S and $(A\Gamma S]$ is a right ideal of S.

Definition 1.5 *[1] Let S be a po-Γ-semigroup. A nonempty subset Q of S is called a quasi-ideal of S if*

(i) $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$,

(ii) *for $x \in Q$ and $y \in S$ such that $y \leq x$ implies $y \in Q$.*

Let S be a *po*-Γ-semigroup. It is easy to show that, if A and B are left (resp. right, quasi-) ideal of S, then $A = (A]$ and $(A \cap B] = (A] \cap (B]$. If L is a left ideal of S and R is a right ideal of S, then $L \cap R$ is a quasi-ideal of S. For each $a \in S$, the right (resp. left) ideal of S generated by a, denoted by $R(a)$ (resp. $L(a)$) is of the form:

$$R(a) = (\{a\} \cup a\Gamma S] = (a] \cup (a\Gamma S] \text{ (resp. } L(a) = (\{a\} \cup S\Gamma a] = (a] \cup (S\Gamma a].$$

2 Main Results

The aim of this section is to generalize Theorem 1.1 and Theorem 1.2 using the concept of *po*-Γ-semigroup.

Theorem 2.1 *Let Q be a nonempty subset of a po-Γ-semigroup S. Then the following are equivalent:*
(1.) Q is a minimal quasi-ideal of S.
(2.) Q is the intersection of a minimal left ideal and a minimal right ideal of S.
(3.) $Q = L(a) \cap R(a)$ for all $a \in Q$.

Proof.(1) $\Rightarrow$ (2). Assume that Q is a minimal quasi-ideal of S. Let $a \in Q$. So $(a\Gamma S]$ is a right ideal of S and $(S\Gamma a]$ is a left ideal of S. It follows that $(a\Gamma S] \cap (S\Gamma a]$ is a quasi-ideal of S. Since $(a\Gamma S] \cap (S\Gamma a] \subseteq (Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$, by minimality of Q, we obtain $(a\Gamma S] \cap (S\Gamma a] = Q$. Next we will show that $(S\Gamma a]$ is a minimal left ideal of S. Let L' be a left ideal of S and $L' \subseteq (S\Gamma a]$. Then $L' \cap (a\Gamma S] \subseteq (a\Gamma S] \cap (S\Gamma a] = Q$. Since $L' \cap (a\Gamma S]$ is a quasi-ideal of S and Q is minimal, $L' \cap (a\Gamma S] = Q$. Thus we have $Q \subseteq L'$. Since $(S\Gamma a] \subseteq (S\Gamma Q] \subseteq (S\Gamma L'] \subseteq L'$, this implies that $L' = (S\Gamma a]$. That is $(S\Gamma a]$ is a minimal left ideal of S. Similarly, $(a\Gamma S]$ is a minimal right ideal. We conclude that Q is the intersection of a minimal left ideal and a minimal right ideal of S.

(2) $\Rightarrow$ (3). Assume that $Q = L' \cap R'$ where L' is a minimal left ideal and R' is a minimal right ideal of S. Let $a \in Q$. Since $a \in L'$ and $a \in R'$, $L(a) \subseteq L'$ and $R(a) \subseteq R'$. Thus $L' = L(a)$ and $R' = R(a)$. Hence, $Q = L(a) \cap R(a)$.

(3) $\Rightarrow$ (1). Assume that $Q = L(a) \cap R(a)$ for all $a \in Q$. Clearly, Q is a quasi-ideal of S. Let Q' be a quasi-ideal of S and $Q' \subseteq Q$. Let $x \in Q'$, then $x \in Q$ and $Q = L(x) \cap R(x)$. Since

$$L(x) \cap R(x) = \{(x] \cup (S\Gamma x]\} \cap \{(x] \cup (x\Gamma S]\} = (x] \cup \{(S\Gamma x] \cap (x\Gamma S]\} \subseteq Q',$$

we have Q is a minimal quasi-ideal of S.

Let S be a *po*-Γ-semigroup. The Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ on S are defined by the following rules:

1. $a\mathcal{L}b$ if and only if $L(a) = L(b)$.
2. $a\mathcal{R}b$ if and only if $R(a) = R(b)$.
3. $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

It is easy to see that $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are equivalence relations on S.

Theorem 2.2 *Let Q be a quasi-ideal of a po-Γ-semigroup S. Then Q is a minimal quasi-ideal of S if and only if Q is a $\mathcal{H}$-class.*

Proof($\Rightarrow$). Assume Q is a minimal quasi-ideal of S. Let $a, b \in Q$. If $a = b$, then $a\mathcal{H}b$. Assume that $a \neq b$. By Theorem 2.1, we have $L(a) \cap R(a) = Q = L(b) \cap R(b)$. Thus $a \in L(b), b \in L(a), b \in R(a)$ and $a \in R(b)$. Thus $L(a) = L(b)$ and $R(a) = R(b)$. Therefore, $a\mathcal{L}b$ and $a\mathcal{R}b$. Hence, $a\mathcal{H}b$.

($\Leftarrow$). Let Q be a $\mathcal{H}$- class. Fix $x \in Q$. Let $y \in Q$. Then $x\mathcal{H}y$. Since $x\mathcal{L}y$ and $x\mathcal{R}y$, $L(x) = L(y)$ and $R(x) = R(y)$. Thus $L(x) \cap R(x) = L(y) \cap R(y)$. Since $y \in L(y) \cap R(y) = L(x) \cap R(x)$ and y is arbitrary, $Q \subseteq L(x) \cap R(x)$. From $x \in Q$ and Q is a quasi-ideal, we obtain

$$L(x) \cap R(x) = \{(x] \cup (S\Gamma x]\} \cap \{(x] \cup (x\Gamma S]\} = (x] \cup \{(S\Gamma x] \cap (x\Gamma S]\} \subseteq Q.$$

Therefore, $Q = L(x) \cap R(x)$. By Theorem 2.1, Q is a minimal quasi-ideal of S.

Definition 2.3 *A po-Γ-semigroup S is said to be a quasi-simple if S is a unique quasi-ideal of S.*

These corollaries follow from Theorem 2.1 and Definition 2.3.

Corollary 2.4 *Let S be a po-Γ-semigroup and Q a quasi-ideal of S. If Q is quasi-simple, then Q is a minimal quasi-ideal of S.*

Corollary 2.5 *Let S be a po-Γ-semigroup. Then S is quasi-simple if and only if S is a $\mathcal{H}$-class.*

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