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Triple Positive Periodic Solutions in Neutral Differential Equations with Variable Parameter

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Abstract

Using Leggett-Williams fixed point theorem we obtain some existence results of three positive periodic solutions for a type of neutral nonlinear differential equation with variable parameter

$$((x(t) - c(t)x(t - \tau))' = rx(t) - f(t, x(t - \tau)).$$

It is worth noting that $c(t)$ is no longer a constant which is different from the corresponding ones of past work.

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1 Introduction

An functional differential equation is neutral if the delayed argument occurs in the highest derivative of the state variable. The field of neutral functional equations (in short NFDEs) is making significant breakthroughs in its practice which are no longer only a specialist's field. The number of NFDEs in the sciences and in applied areas has been increasing tremendously, where they continue to play a crucial role. Applications range from population models to neutral networks, from blood cell models to mechanics models, and from studies of engines to the theory of business cycles. For more details, see [1]-[2].

The purpose of this paper is to investigate the existence of positive periodic solutions to the nonlinear neutral differential equation with variable parameter

$$(x(t) - c(t)x(t - \tau))' = rx(t) - f(t, x(t - \tau)), \tag{1.1}$$

where $f(t, x) \in C(R \times R^+, R^+)$, $R^+ = (0, +\infty)$ and $f(t + T, \cdot) = f(t, \cdot)$; $c(t)$ is a $T-$periodic function; r, τ and T are given constants with $T > 0$.

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2 Main Lemmas

For the sake of convenience, we list the following conditions which will be needed in our study of Eq. (1.1).
(H1) $r > 0$; $k = e^{-rT},\ 0 < k < 1$;
(H2) $c_0 \in [0, \frac{k}{k+1}),\ c_0 = \max_{t\in[0,T]} |c(t)|$;
(H3) there exist two positive constants $m,\ M$ such that

$$0 < m \leq f(t,u) \leq M < \infty, \quad \forall (t,u) \in R \times (0,+\infty).$$

Let P be a cone and $a, b, r > 0$ be constants. Set

$$P_r = \{u \in P : \| u \| < r\}, \quad P(\rho, a, b) = \{u \in P : \rho(u) \geq a, \| u \| \leq b\}.$$

Lemma 2.1 *(Leggett-Williams fixed-point theorem [4]). Let $\widetilde{T} : \bar{P}_h \to \bar{P}_h$ be a completely continuous map and ρ be a nonnegative continuous concave functional on P such that $\rho(u) \leq \| u \|$,*
$\forall u \in \bar{P}_h$. Suppose there exist $a,\ b,\ d$ with $0 < a < b < d \leq h$, such that
(i) $\{u \in P(\rho, b, d) : \rho(u) > b\} \neq \emptyset$ and $\rho(\widetilde{T}u) > b$ for all $u \in P(\rho, b, d)$;
(ii) $\| \widetilde{T}u \| < a$ for all $u \in \bar{P}_a$;
(iii) $\rho(\widetilde{T}u) > b$ for all $u \in P(\rho, b, h)$ with $\| \widetilde{T}u \| > d$.
Then $\widetilde{T}$ has at least three fixed points u_1, u_2 and u_3 satisfying

$$\| u_1 \| < a, \quad b < \rho(u_2), \quad \| u_3 \| > a, \quad \rho(u_3) < b.$$

Let

$$C_T = \{x | x \in C(R,R),\ x(t+T) \equiv x(t),\ \forall t \in R\}$$

with the norm

$$||\varphi|| = |\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \quad \forall \varphi \in C_T.$$

Clearly, C_T is a Banach space. Define linear operator:

$$A : C_T \to C_T, \quad [Ax](t) = x(t) - c(t)x(t-\tau), \quad \forall t \in R.$$

Lemma 2.2 *[3] If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(t-(i-1)\tau) f(t-j\tau),\ c_0 < 1,\ \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum_{j=1}^{\infty} \prod_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t+j\tau+\tau),\ \sigma > 1,\ \forall f \in C_T. \end{cases}$$

(2)

$$\int_0^T |[A^{-1}f](t)|dt \le \begin{cases} \frac{1}{1-c_0}\int_0^T |f(t)|dt, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1}\int_0^T |f(t)|dt, \ \sigma > 1, \ \forall f \in C_T. \end{cases}$$

(3)

$$||A^{-1}f|| \le \begin{cases} \frac{||f||}{1-c_0}, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{||f||}{\sigma-1}, \ \sigma > 1, \ \forall f \in C_T. \end{cases}$$

Now, we define a cone P in C_T by

$$P = \{u(t) \in C_T : u(t) \ge k||u||\}.$$

From $c_0 \in [0, \frac{k}{k+1})$, if $f \in P$, by Lemma 2.2 we have

$$\begin{aligned} [A^{-1}f](t) = \ & f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau) \\ & \ge k||f|| - \sum_{j=1}^{\infty} c_0^j||f|| \\ & = k||f|| - \frac{c_0}{1-c_0}||f|| = \frac{k-kc_0-c_0}{1-c_0}||f|| := \alpha||f||. \end{aligned} \tag{2.1}$$

Consider the following equation:

$$x'(t) = r(A^{-1}x)(t) - f(t,(A^{-1}x)(t-\tau)). \tag{2.2}$$

We have the following result.

Lemma 2.3 *$x(t)$ is a $T-$periodic solution of Eq. (2.2) if only if $(A^{-1}x)(t)$ is a $T-$periodic solution of Eq. (1.1).*

From $[Ax](t) = x(t) - c(t)x(t-\tau)$ and Lemma 2.2, we have

$$(A^{-1}x)(t) = x(t) + c(t)(A^{-1}x)(t-\tau).$$

Now we transform Eq. (2 .2) to the form:

$$x'(t) = rx(t) + rc(t)(A^{-1}x)(t-\tau) - f(t,(A^{-1}x)(t-\tau)). \tag{2.3}$$

Lemma 2.4 *If $x(t) \in P$, then $x(t)$ is a solution of Eq. (2.3) if only if*

$$x(t) = \int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds,$$

here $K(t,s) = \frac{e^{-r(s-t)}}{1-e^{-rT}}, \ t \in R.$

Proof. Let $x(t) \in P$ be a solution of Eq. (2.3). We multiply both sides of Eq. (2.3) with e^{-rt} and then integrate from t to $t+T$ to obtain

$$x(t+T)e^{-r(t+T)} - x(t)e^{-rt} = \int_t^{t+T} K(t,s)[rc(s)(A^{-1}x)(s-\tau) - f(s,(A^{-1}x)(s-\tau))]ds.$$

Using the fact that $x(t+T) = x(t)$, the above expression can be put in the form

$$x(t) = \int_t^{t+T} \frac{e^{-r(s-t)}}{1-e^{-rT}}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds.$$

This completes the proof.

Hence we define an operator $\widetilde{T} : P \to C_T$ by

$$(\widetilde{T}x)(t) = \int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds,$$

here $K(t,s) = \frac{e^{-r(s-t)}}{1-e^{-rT}},\ t \in R$. Clearly, we have

$$\frac{e^{-rT}}{1-e^{-rT}} \le K(t,s) \le \frac{1}{1-e^{-rT}} \tag{2.4}$$

and the existence of a $T-$periodic solution of Eq. (2.2) is equivalent to the existence of a fixed point of the operator $\widetilde{T}$.

Lemma 2.5 *Suppose the assumptions $(H1)-(H3)$ hold, then $\widetilde{T}P \subset P$.*

Proof. $\forall x \in P$, *by (2.4) we have*

$$\begin{aligned}
&(\widetilde{T}x)(t+T) = \textstyle\int_{t+T}^{t+2T} K(t+T,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&= \textstyle\int_t^{t+T} K(t+T,s+T)[f(s+T,(A^{-1}x)(s+T-\tau)) - rc(s+T)(A^{-1}x)(s+T-\tau)]ds \\
&= \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&= (\widetilde{T}x)(t)
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{T}x)(t) &= \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&\ge \tfrac{e^{-rT}}{1-e^{-rT}} \textstyle\int_t^{t+T}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&= k \textstyle\int_t^{t+T} \tfrac{1}{1-e^{-rT}}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&\ge k \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds \\
&= k||\widetilde{T}||.
\end{aligned}$$

Hence $\widetilde{T}P \subset P$.

At the same time, if $(H1)-(H3)$ hold, we claim: $x(t)$ is a positive fixed point of $\widetilde{T}$ if and only if $(A^{-1}x)(t)$ is a positive periodic solution of Eq. (1.1). In fact, if $x(t)$ is a positive fixed point of $\widetilde{T}$, by Lemma 2.3, $x(t)$ is a periotic solution of Eq. (2.2) and $(A^{-1}x)(t)$ is a periodic solution of Eq. (1.1). Since $x \in P$, by (2.1) we have $(A^{-1}x)(t) > 0$. Hence $(A^{-1}x)(t)$ is a positive periodic solution of Eq. (1.1). The inverse is true.

3 Three positive periodic solution for Eq. (1.1)

Define the nonnegative continuous concave functional $\rho : P \to [0, \infty)$ by

$$\rho(u) = \min_{0\le t\le T} u(t), \quad u \in P.$$

Theorem 3.1 *Suppose that assumptions (H1)-(H3) hold and there exist a, b, h, d satisfying*

$$0 < a < b \le k^2 d < d \le h.$$

If the following condition holds: $e^{-rT} - rc_0\alpha e^{-rT}T < 1$. *Then Eq. (1.1) has at least triple positive periodic solutions* x_1, x_2 *and* x_3 *satisfying*

$$\| x_1 \| < a, \quad b < \rho(x_2), \quad \| x_3 \| > a, \quad \rho(x_3) < b.$$

Proof. Define a mapping $\widetilde{T} : P \to P$ by

$$(\widetilde{T}x)(t) = \int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds.$$

For $\forall x \in \bar{P}_h$, by (2.4) and Lemma 2.2, we have

$$\begin{aligned}(\widetilde{T}x)(t) &= \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\le \tfrac{1}{1-e^{-rT}} \textstyle\int_t^{t+T}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\le \tfrac{T}{1-e^{-rT}}(M - \tfrac{rc_0 h}{1-c_0}) = h,\end{aligned}$$

where $h = \frac{TM(1-c_0)}{(1-e^{-rT})(1-c_0)+rc_0T}$. Similarly, $\widetilde{T}x \in \bar{P}_a$ for all $x \in \bar{P}_a$. Now we show

$$\{x \in P(\rho,b,d) : \rho(x) > b\} \ne \emptyset \ and\ \rho(\widetilde{T}x) > b,\ for\ x \in P(\rho,b,d).$$

In fact, set $x = \frac{b+d}{2}$, $||x|| = \frac{b+d}{2} < d$ and $\rho(x) > b$. So $\{x \in P(\rho,b,d) : \rho(x) > b\} \ne \emptyset$. On the other hand, $\forall x \in P(\rho,b,d)$, we have $b \le ||x|| \le d$ which implies, by (2.1), $(A^{-1}x)(t) \ge \alpha||x|| \ge \alpha b$. Hence, by $e^{-rT} - rc_0\alpha e^{-rT}T < 1$ we get

$$\begin{aligned}\rho(\widetilde{T}x) &= \min_{0\le t\le T}(\widetilde{T}x)(t)\\ &= \min_{0\le t\le T} \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\ge \tfrac{e^{-rT}}{1-e^{-rT}} \min_{0\le t\le T} \textstyle\int_t^{t+T}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\ge \tfrac{e^{-rT}T}{1-e^{-rT}}[m - rc_0\alpha b] = b,\end{aligned}$$

where $b = \frac{e^{-rT}Tm}{1-e^{-rT}+rc_0\alpha e^{-rT}T}$. Finally, we prove that the condition (3) of Lemma 2.1 holds. If $x \in P(\rho,b,h)$ and $||\widetilde{T}x|| > d$, then $\widetilde{T}x \in P$, we have

$$\begin{aligned}\rho(\widetilde{T}x) &= \min_{0\le t\le T} \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\ge \tfrac{e^{-rT}}{1-e^{-rT}} \min_{0\le t\le T} \textstyle\int_t^{t+T}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &= k \min_{0\le t\le T} \textstyle\int_t^{t+T} \tfrac{1}{1-e^{-rT}}[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\ge k \min_{0\le t\le T} \textstyle\int_t^{t+T} K(t,s)[f(s,(A^{-1}x)(s-\tau)) - rc(s)(A^{-1}x)(s-\tau)]ds\\ &\ge k^2||\widetilde{T}x|| > k^2 d \ge b.\end{aligned}$$

Hence, conditions of Lemma 3.1 are all satisfied. We complete the proof. As an application, we consider the following NFDE:

Example 3.2

$$(u(t) - \frac{1}{20}\sin^2 t u(t-\pi))' = \frac{1}{\pi}u(t) - \frac{\cos^2 t}{1+\sin^2 u(t-\pi)}, \tag{3.1}$$

where

$$c(t) = \frac{1}{20}\sin^2 t,\ r = \frac{1}{\pi},\ f(t,u) = \frac{\cos^2 t}{1+\sin^2 u(t-\pi)},\ T = \pi,\ c_0 = \frac{1}{20},\ \alpha \doteq 0.3174.$$

From simple calculation, we have

$$e^{-rT} - rc_0\alpha e^{-rT}T \doteq 0.3613 < 1.$$

Applying Theorem 3.1, Eq. (3.1) has at least three positive 2π-periodic solution.

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