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On One-Sided Prime Ideals of b-Semirings

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Abstract

This paper contains some results on one-sided prime ideals in b-semirings analogous to commutative and largely non-commutative ring(semiring) theory.

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1 Introduction

The concept of b-semirings was introduced by Ronnason [7]. In this paper, we investigate and obtain some results on one-sided prime ideals in b-semirings. Unless otherwise stated, throughout this paper, all b-semirings shall mean non-commutative b-semirings since one-sided ideals coincide in the commutative case. For the sake of completeness and clarity, we introduce and quote here some definitions and examples used throughout this paper. To show some interplay between a semiring and a b-semiring, we start. A semiring is a nonempty set with two associative binary operations '+' and '.' with an additive identity called zero of the semiring such that '.' distributes over '+' and $a.0 = 0.a = 0$ for every a of the semiring.

Definition 1.1. [7] *Let S be a non-empty set and $*_1$ and $*_2$ be two binary operations on S. Then $(S, *_1, *_2)$ is called a b-semiring if $(S, *_1)$ and $(S, *_2)$ are semigroups and for all $a, b, c \in S$, $a *_1 (b *_2 c) = (a *_1 b) *_2 (a *_1 c)$, $(b *_2 c) *_1 a = (b *_1 a) *_2 (c *_1 a)$, $a *_2 (b *_1 c) = (a *_2 b) *_1 (a *_2 c)$, and*

$(b *_1 c) *_2 a = (b *_2 a) *_1 (c *_2 a)$.

A b-semiring is a restricted class of semiring. For a detail of semiring, one can refer [8].

Definition 1.2. *. A b-semiring S is said to be commutative if* $a *_1 b = b *_1 a$ *and* $a *_2 b = b *_2 a$.

Example 1.3. [7] *Let* $(S, .)$ *be a semigroup with zero* 0. *Define a binary operation* o *on* S *by* $xoy = 0$ *for all* $x, y \in S$. *Then* $(S, ., o)$ *is a commutative b-semiring.*

Definition 1.4. *Let* $(S, *_1, *_2)$ *be a b-semiring. An element* $e \in S$ *is called left neutral in* $(S, *_1, *_2)$ *if* $e *_1 a = a$ *and* $e *_2 a = a$. *Right neutral element is defined correspondingly. Further, if it is both left as well as right neutral, then we call it neutral element or identity of* $(S, *_1, *_2)$. *For brevity, we emphasize that e is a neutral element if it is neutral with respect to both the operations* $*_1$ *and* $*_2$

Definition 1.5. *Let* $(S, *_1, *_2)$ *be a b-semiring. An element* $0 \in S$ *is called left absorbing in* $(S, *_1, *_2)$ *if* $0 *_1 a = 0$ *and* $0 *_2 a = 0$. *Right absorbing element is defined correspondingly. An element that is both left absorbing as well as right absorbing is called absorbing element or zero of* $(S, *_1, *_2)$.

Example 1.6. [7] *Let* $(S, *, *)$ *be a left zero b-semiring, that is,* $a * b = a$ *for all* $a, b \in S$. *We further note that this is a non-commutative b-semiring and each element* $n \in S$ *is right neutral and left absorbing in* S.

Definition 1.7. [7] *Let* $(S, *_1, *_2)$ *be a b-semiring. A non-empty subset* T *of* S *is called a subb-semiring if* $(T, *_1, *_2)$ *is a b-semiring. In other words,* T *is a subb-semiring iff* $a *_1 b \in T$ *and* $a *_2 b \in T$ *for all* $a, b \in T$.

Example 1.8. *Let* R *be the set of real numbers. Let the binary operations be defined as (i)* $a + b = 0$ *and* $a.b = a$, *for all* $a, b \in R$. *(ii)* $a + b = 0$ *and* $a.b = b$, *for all* $a, b \in R$. *Then* $(R, +, .)$ *is a b-semiring in each of the two cases above. Here 0 is the absorbing zero or simply zero of the b-semiring* $(R, +, .)$.

2 Prime Right Ideals of b-semirings

Ideals and prime ideals are defined for semirings in the same way as for rings. N. H. McCoy [4] has defined that a two sided ideal T in a ring R is a prime ideal provided that if $AB \subseteq T$, A, B are right ideals, then either $A \subseteq T$ or $B \subseteq T$. In [2] K. Koh has defined that a right (left) ideal I in a ring R is of prime type if $I \neq R$ and $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$, where

A and B are the right(left) ideals of R. In [1] F. Hansen called these ideals prime right ideals. In the case of commutative rings, an ideal of a prime type and a prime ideal coincide.

Definition 2.1. *A right ideal P of a b-semiring S is called a prime right ideal if for every right ideals I, J of S, $I *_1 J \subseteq P$, $I *_2 J \subseteq P$ imply $I \subseteq P$ or $J \subseteq P$.*

Definition 2.2. *A proper right (resp. left) ideal M of S is said to be a maximal right (resp. left) ideal of S, if whenever B is a right (resp. left) ideal of S and $M \subseteq B \subseteq S$, then $B = M$ or $B = S$, that is, the only ideal that properly contains a maximal ideal is the entire b-semiring*

Example 2.3. *Let $(R, +, .)$ be a b-semiring as in the Example [1.8(i)]. Let Z be a set of integers. Then Z is a right ideal but not a left ideal of R. Moreover, let W be the set of whole numbers and Z^- be the set of negative integers including zero. Then we have that both are maximal right ideals but not maximal left ideals. Similarly, we can construct maximal left ideals but not right ideals by considering Example [1.8(ii)].*

Theorem 2.4. *The following assertions are equivalent for a b-semiring S*
(i) If (a), (b) are principal one-sided ideals generated by a, $(a) *_1 (b) \subset P$ and $(a) *_2 (b) \subset P$, then $a \in P$ or $b \in P$.
(ii) $a *_1 R *_1 b \subset P$ and $a *_2 R *_2 b \subset P$ implies $a \in P$ or $b \in P$.
(iii) P is a prime right ideal.
(iv) If I_1,I_2 are right ideals and $I_1 *_1 I_2 \subset P$ and $I_1 *_2 I_2 \subset P$, then $I_1 \subset P$ or $I_2 \subset P$.
(v) If J_1,J_2 are right ideals, $J_1 *_1 J_2 \subset P$ and $J_1 *_2 J_2 \subset P$, then $J_1 \subset P$ or $J_2 \subset P$.

N. H. McCoy [5] proved the Theorem [2.4] in the case of rings. We prove the same for the b-semiring S.

Proof. *It is clear that (i) implies (ii). To prove that (ii) implies (iii), let $a *_1 S *_1 b \subset P$ and $a *_2 S *_2 b \subset P$, then $S *_1 a *_1 S *_1 b *_1 S \subset P$ and $S *_2 a *_2 S *_2 b *_2 S \subset P$, and so we have $(a) *_1 (a) *_1 (b) *_1 (b) *_1 (b) \subset P$ and $(a) *_2 (a) *_2 (b) *_2 (b) *_2 (b) \subset P$. This shows $a \in P$ or $b \in P$. To show that (iii) implies (iv), suppose $I_1 *_1 I_2 \subset P$ and $I_1 *_2 I_2 \subset P$ for right ideals I_1, I_2 and let $I_1 \nsubseteq P$. There exists an element I_1 not in P. Thus, for every element b of I_2, $a *_1 S *_1 b \subset I_1 *_1 I_2 \subset P$ and $a *_2 S *_2 b \subset I_1 *_2 I_2 \subset P$. Therefore, from (iii), $b \in P$ and this proves $I_2 \subset P$. Similarly, we can show that (iii) implies (v). Trivially, (iv) or (v) implies (i).*

To prove proposition 2.6, we use the following proposition, which is obvious by definition.

Proposition 2.5. *Let U be a right ideal of a b-semiring S. Then*
(i) U is a prime right ideal, and
(ii) If $a, b \in S$ such that $a *_1 S *_1 b \subseteq U$, then $a \in S$ or $b \in S$ and $a *_2 S *_2 b \subseteq U$, then $a \in S$ or $b \in S$.

Proposition 2.6. *Any maximal right ideal of a b-semiring S with identity e is a prime right ideal.*

Proof. *We assume that I is a maximal right ideal of a b-semiring S and $a *_1 S *_1 b \subseteq I$. If a is not in I, then we show that $b \in I$. The maximality of I implies that right ideal generated by I and a must be the whole b-semiring S, i.e., $S = I *_1 a *_2 S$. Hence, there exists $i \in I$ and $r_0 \in S$ such that $e = i *_1 a *_2 r_0$. Now $b = e *_2 b = (i *_1 a *_1 r_0) *_2 b = i *_2 b *_1 a *_2 r_0 *_2 b \in I$. Thus, I is a prime right ideal, which proves that every maximal right ideal of a b-semiring S is a prime right ideal. Similarly, one can show the above proposition by taking the second operation.*

Definition 2.7. *A right ideal I of a b-semiring S is called semiprime right ideal if and only if for any right ideal H of S, $H *_1 H \subseteq I$ implies $H \subseteq I$ and $H *_2 H \subseteq I$ implies $H \subseteq I$. Clearly, every prime right ideal of a b-semiring S is a semiprime right ideal of S.*

L. Fuchs [3] introduced a notion of strongly irreducible ideals and he calls it primitive. R. L. Blair [6] introduced the terminology strongly irreducible.

Definition 2.8. *A right ideal I of a b-semiring S is called an irreducible (strongly irreducible) right ideal if $J \cap K = I (J \cap K \subseteq I)$ implies either $J = I$ or $K = I (J \subseteq I$ or $K \subseteq I)$ for every right ideal J and K of S.*

Proposition 2.9. *Let I be a right ideal of a b-semiring S. If a is not in I, then there exists an irreducible right ideal containing I and not containing a.*
Proof: If $\{A_i : i \in \Omega\}$ is a chain of right ideals of S containing I and not containing a, then $\cup A_i$ is a right ideal of S containing I and not containing a. Therefore, by Zorn's Lemma, the set of all right ideals of S containing I and not containing a has a maximal element A. Suppose $A = B \cap C$, where B and C are both right ideals of S properly containing A. Then by the choice of A, $a \in B$ and $a \in C$. Then $a \in B \cap C = A$, which is a contradiction. Hence A is an irreducible right ideal of the b-semiring S.

Proposition 2.10. *Any right ideal I of a b-semiring S is the intersection of all the irreducible right ideals of S containing I.*
Proof. Let I be a right ideal of a b-semiring S and $\{A_i : i \in \Omega\}$ be the collection of irreducible right ideals of S containing I, then $I \subseteq \cap A_i$. For the reverse inclusion, let x is not in I, then by proposition 2.9, there exists an irreducible right ideal A of S containing I but not containing x. Thus x is not in $\cap A_i$, hence $I = \cap A_i$.

Theorem 2.11. *Let $(S, *_1, *_2)$ be a b-semiring.Then the following assertions are true.*
(i) Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of all prime left ideals of S. If $\cap_{\alpha\in I} L_\alpha \neq \phi$, then $\cap_{\alpha\in I} L_\alpha$ is a prime left ideal of S.

(ii) Let $\{R_\alpha \mid \alpha \in I\}$ be a collection of all prime right ideals of S. If $\cap_{\alpha\in I} R_\alpha \neq \phi$, then $\cap_{\alpha\in I} R_\alpha$ is a prime right ideal of S.
Proof(i): Let $a \in \cap_{\alpha\in I} L_\alpha$ and $s \in S$. Then $a \in L_\alpha$ for all $\alpha \in I$. This shows that $s *_1 a \in L_\alpha$ and $s *_2 a \in L_\alpha$, for all $\alpha \in I$. Hence $s *_1 a \in \cap_{\alpha\in I} L_\alpha$ and $s *_2 a \in \cap_{\alpha\in I} L_\alpha$. The proof of (ii) is similar to the proof of (i).

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