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A Characterization of a Toeplitz Operator on Bergman and Hardy Spaces

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Abstract

We study the cyclicity of the inverse shift operator on both the Bergman and Hardy spaces over a crescent region.

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Introduction and main result

Recall that a crescent G is a bounded region of the complex plane $\mathbf{C}$ of the form: $\Omega\backslash\overline{W}$, where Ω and W are Jordan regions, $W \subseteq \Omega$, and ∂W and $\partial\Omega$ have precisely one point in common which is called the *multiple boundary point* of G. For our purposes there is no loss to assume (which we do) that $0 \in W$ and G has multiple boundary point 1. In this paper we restrict our attention to crescents G of the form $\Omega\backslash\overline{W}$, where ∂W and $\partial\Omega$ are differentiable except (maybe) at 1 and are of bounded curvature except (maybe) at 1. Under these circumstances there are four angles at 1 (which we call α, β, γ and δ), subtended by G and the components of $\mathbf{C}\backslash\overline{\mathbf{G}}$;(see Figure 1).

An operator T on a Banach space$\Im$ is said to be cyclic if there exists h in $\Im$ such that $\{p(T)h : p \text{ is a polynomial}\}$ is dense in $\Im$. In this case h is called a cyclic vector for T in $\Im$.

For $1 \le t < \infty$, the Bergman space $L^t_a(G) = \{f : f \text{is analytic in } G\}$ and $\int_G |f|^t \, dA < \infty$, and the classical Hardy space $H^t(D) = \{f : f \text{ is analytic in } D\}$ and

$$\sup_{0<r<1} \int_{\partial D} |f(r\xi)|^t \, dm(\xi) < \infty.$$

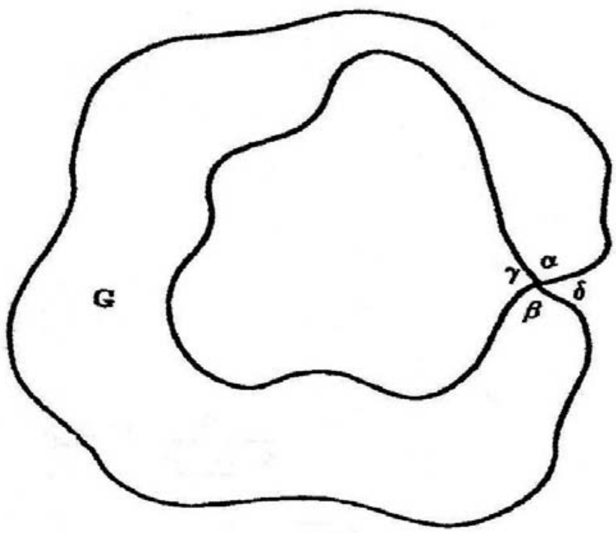

Figure 1

Now the shift operator M_z, defined by $M_z(f) = zf$ is cyclic on $L^t_a(G)$ (resp. $H^t(G)$), if there exists h in $L^t_a(G)$ (resp. $H^t(G)$) such that $\{hp : p \text{ is a polynomial}\}$ is dense in $L^t_a(G)$ (resp. $H^t(G)$). We state the following result that is established in [2, Theorem 4.11]

Theorem 1 *If M_z is cyclic on $H^t(E)$, then there is a bounded cyclic vector for M_z on $L^t_a(E)$; for any bounded simply connected region E*

Whether or not the converse of 1 holds remains an open question. Since $0 \in W$, where W is the bounded component of $\mathbf{C} \setminus \overline{\mathbf{G}}$, M_z is invertible on $H^t(G)$ and $L^t_a(G)$ and has inverse $M_{\frac{1}{z}}$. Observe that $M_{\frac{1}{z}}$ on the Bergman space $L^t_a(G)$ (resp. $H^t(G)$) is cyclic if $\{p(\frac{1}{z})h : p \text{ is a polynomial}\}$ is dense in $L^t_a(G)$ (resp.$H^t(G)$. Now it follows from the work in [4] along with [1], that, in the case $\alpha = \beta$, M_z is cyclic on both the Hardy and Bergman spaces of G (for $t = 2$) if and only if $\gamma > \alpha (= \beta)$. In [8], the authors proved a more general result for "non-symmetric" crescent regions G in the context of the Hardy space. They showed that M_z on $H^2(G)$ is cyclic if and only if $\gamma > \min(\alpha, \beta)$

(cf. [8]). Using 1 and an argument similar to the proof of [4, Theorem 4.1], we see that the result of [8] carries over to the setting of the Bergman space. Furthermore, slight variants of the proofs in the aforementioned literature hold in the context of the Hardy space for any t, $1 \leq t < \infty$. Again, using Theorem 1 and an argument similar to the proof of [4, Theorem 4.1], we can extend once again to the Bergman space setting. We summarize in the following theorem.

Theorem 2 *If G is as above, then M_z on $L^t_a(G)$ and $H^t(G)$ is cyclic $(1 \leq t < \infty)$ if and only if $\beta > \min(\alpha, \gamma)$*

The next theorem amounts to an observation based on the above theorem.

Theorem 3 *In either $L^t_a(G)$ or $H^t(G)$ and for $1 \leq t < \infty$: $M_z^{-1} = M_{\frac{1}{z}}$ is cyclic if and only if $\delta > \min(\alpha, \beta)$*

Proof. We are assuming that G is as described in the beginning of this paper. Let $G^* = \psi(G)$, where $\psi(z) := \frac{1}{z}$. Applying Theorem 3 to M_z on $L^t_a(G^*)$ and $H^t(G^*)$, and then composing with ψ gives the result.

Corollary 1 *If G is as above, then M_z and M_z^{-1} are both cyclic on $L^t_a(G)$ and $H^t(G)$ $(1 \leq t < \infty)$ if and only if $\min(\gamma, \delta) > \min(\alpha, \beta)$*

Theorem 4 *Let G be as above, and let φ be a conformal mapping from D one-to-one and onto G.Then the Toeplitz operator T_φ is cyclic on $H^2(D)$ if and only if $\gamma > \min(\alpha, \beta)$ and $T_\varphi^{-1} = T_{\frac{1}{\varphi}}$ is cyclic if and only if $\delta > \min(\alpha, \beta)$.*

Proof. By Theorem 3, the Toeplitz operator T_φ is cyclic on $H^2(D)$ if and only if $\gamma > \min(\alpha, \beta)$ and $T_\varphi^{-1} = T_{\frac{1}{\varphi}}$ is cyclic if and only if $\delta > \min(\alpha, \beta)$. In fact the same result holds for T_φ on $L^2_a(D)$. This is established by observing that the conclusion of Theorem 3 holds for M_z on $L^2_a(\eta) := \{f : f \text{ is analytic in } G\}$ and $\int_G |f|^2 |(\varphi^{-1})'|^2 dA < \infty$. Indeed, under our assumptions concerning G, ${\varphi^{-1}}' \in L^\infty(G)$ and hence applying Theorem 1 gives one direction. The converse is a straightforward argument similar to the proof of [4, Theorem 4.1] and uses elementary estimates of ${\varphi^{-1}}'$

Definition 1 *A Caratheodory region is an open connected subset of C whose boundary equal its outer boundary.*

The next theorem is a combination of results from [5] and [7].

Theorem 5 *If G is a bounded Caratheodory region, then 1 is a cyclic vector for the shift operator on $H^2(G)$.*

We end with the following example:

Example Let G be a ribbon shaped region that spiral out to the unit circle and such that $0 \notin \overline{G}$, then $M_{\frac{1}{z}}$ on $H^2(G)$ is cyclic and in fact 1 is a cyclic vector irregardless how thick or thin G is, or how quickly G spirals. This follows because $G^* := \{z : \frac{1}{z}\}$ is in G is a so-called cornucopia (Figure 2) and hence a Caratheodory region [6, Prop. 8.10] and by Theorem 5, polynomials are dense in $H^2(G^*)$. However, whether or not M_z on $H^2(G)$ is cyclic does depend on the thickness of G and on its rate of spiraling [3, Remark 6].

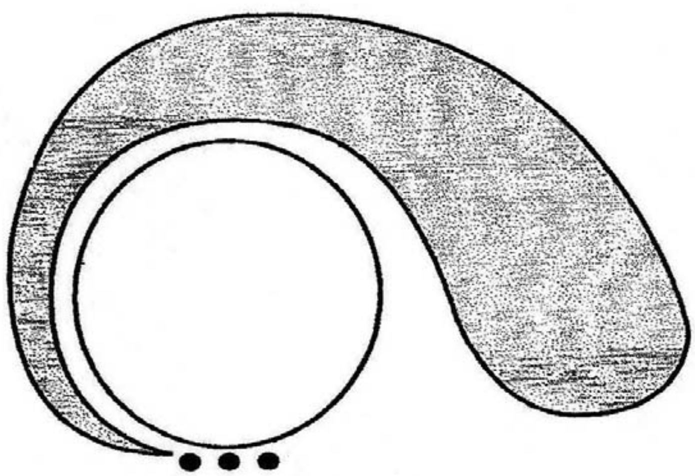

Figure 2

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