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S-iterative Process for a Pair of Single Valued and Multi-valued Nonexpansive Mappings

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Abstract

Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that $Fix(t) \cap Fix(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Suppose $\{x_n\}$ is generated iterative by $x_1 \in E$,

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n t y_n, \ \forall n \geq 1, \end{aligned}$$

where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

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1 Introduction

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E and by

$KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}, \quad A,B \in FB(X),$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B.

A mapping $t : E \to E$ is said to be *nonexpansive* if

$$\|tx - ty\| \le \|x - y\| \text{ for all } x, y \in E.$$

A point x is called a fixed point of t if $tx = x$.

A multi-valued mapping $T : E \to FB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \le \|x - y\| \text{ for all } x, y \in E.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$.

We use the notation $Fix(T)$ stands for the set of fixed points of a mapping T and $Fix(t) \cap Fix(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

In 2006, S. Dhompongsa et al. [5] proved a common fixed point theorem for two nonexpansive commuting mappings.

Theorem 1.1 *(see [5], Theorem 4.2) Let E be a nonempty bounded closed convex subset of a uniformly Banach space X, $t : E \to E$, and $T : E \to KC(E)$ a nonexpansive mapping and a multi-valued nonexpansive mapping respectively. Assume that t and T are commuting , i.e. if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds $tx \in Tty$. Then t and T have a common fixed point.*

The purpose of this paper is to study the new iterative process, called the modified S-iteration method with respect to a pair of single valued and multi-valued nonexpansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

2 Preliminary Notes

The important property of a uniformly convex Banach space we use is the following lemma proved by Schu [2] in 1991.

Lemma 2.1 *(see [2]) Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \le u_n \le c < 1$ for all $n \ge 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$ and $\lim_{n\to\infty} \|u_n x_n + (1-u_n)y_n\| = a$ for some $a \ge 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following observation will be used in proving our results and the proof is a straightforward.

Lemma 2.2 *Let X be a Banach space and E be a nonempty closed convex subset of X. Then,*

$$dist(y,Ty) \le \|y-x\| + dist(x,Tx) + H(Tx,Ty),$$

where $x, y \in E$ and T is a multi-valued nonexpansive mapping from E into $FB(E)$.

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle. A mapping t defined on a subset E of a Banach space X is said to be demiclosed if any sequence $\{x_n\}$ in E the following implication holds: $x_n \rightharpoonup x$ and $tx_n \to y$ implies $tx = y$.

Theorem 2.3 *(see [1]) Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $t : E \to E$ be a nonexpansive mapping. If a sequence $\{x_n\}$ in E converges weakly to p and $\{x_n - tx_n\}$ converges to 0 as $n \to \infty$, then $p \in Fix(t)$.*

In 2009, Agarwal et al. [4] introduced the S-iteration following well-known iteration.

For E a convex subset of a linear space X and t a mapping of E into itself, the iterative sequence $\{x_n\}$ of the S-iteration process is generated from $x_1 \in E$ and is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n t x_n, \\ x_{n+1} &= (1-\alpha_n)tx_n + \alpha_n t y_n, \quad \forall n \ge 1, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in (0, 1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

In 2010, Sokhuma and Keawkhao [3] defined the modified Ishikawa iteration method scheme for a pair of single valued and multi-valued nonexpansive mappings as follows:

Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n, \quad \forall n \geq 1, \end{aligned}$$

where $x_1 \in E, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

They proved the strong convergence theorem of a sequence from this process in a nonempty compact convex subset of a uniformly convex Banach space.

In this paper, we present a new iteration method modifying the above ones and call it the modified S-iteration.

Definition 2.4 *Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified S-iteration is defined by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n t y_n, \quad \forall n \geq 1, \end{aligned} \tag{1}$$

where $x_1 \in E, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

3 Main Results

In this section, we present our main results.

Lemma 3.1 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1). Then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_1 \in E$ and $w \in Fix(t) \cap Fix(T)$, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)z_n + \alpha_n t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &\leq (1-\alpha_n)\|z_n - w\| + \alpha_n \|t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &\leq (1-\alpha_n)\|z_n - w\| + \alpha_n \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &\leq (1-\alpha_n)\|z_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|z_n - w\| \\ &= (1-\alpha_n)dist(z_n, Tw) + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n dist(z_n, Tw) \\ &\leq (1-\alpha_n)H(Tx_n, Tw) + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n H(Tx_n, Tw) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned}$$

Since $\{\|x_n - w\|\}$ is a decreasing and bounded sequence, we can conclude that the limit of $\{\|x_n - w\|\}$ exists.

We can see how Lemma 2.1 is useful via the following lemma.

Lemma 3.2 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1). If $0 < a \leq \alpha_n \leq b < 1$ for some a, b in $\mathbb{R}$, then $\lim_{n\to\infty} \|ty_n - z_n\| = 0$.*

Proof. Let $w \in Fix(t) \cap Fix(T)$. By Lemma 3.1, we put $\lim_{n\to\infty} \|x_n - w\| = c$ and consider

$$\begin{aligned}
\|ty_n - w\| &\leq \|y_n - w\| \\
&= \|(1-\beta_n)x_n + \beta_n z_n - w\| \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|z_n - w\| \\
&= (1-\beta_n)\|x_n - w\| + \beta_n dist(z_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned}$$

So, we have

$$\limsup_{n\to\infty} \|ty_n - w\| \leq \limsup_{n\to\infty} \|y_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c. \tag{2}$$

Recall that

$$\begin{aligned}
\|z_n - w\| &= dist(z_n, Tw) \\
&\leq H(Tx_n, Tw) \\
&\leq \|x_n - w\|.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \|z_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c.$$

Thus,

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - w\| \\
&= \lim_{n\to\infty} \|(1-\alpha_n)z_n + \alpha_n ty_n - w\| \\
&= \lim_{n\to\infty} \|\alpha_n ty_n - \alpha_n w + z_n - \alpha_n z_n + \alpha_n w - w\| \\
&= \lim_{n\to\infty} \|\alpha_n(ty_n - w) + (1-\alpha_n)(z_n - w)\|.
\end{aligned}$$

By Lemma 2.1, we can conclude that

$$\lim_{n\to\infty} \|(ty_n - w) - (z_n - w)\| = \lim_{n\to\infty} \|ty_n - z_n\| = 0.$$

The following lemmas are useful and crucial for our main results.

Lemma 3.3 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. Let $w \in Fix(t) \cap Fix(T)$. We put, as in Lemma 3.2, $\lim_{n\to\infty} \|x_n - w\| = c$. For $n \geq 1$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1-\alpha_n)z_n + \alpha_n ty_n - w\| \\
&\leq (1-\alpha_n)\|z_n - w\| + \alpha_n \|ty_n - w\| \\
&= (1-\alpha_n)dist(z_n, Tw) + \alpha_n \|ty_n - w\| \\
&\leq (1-\alpha_n)H(Tx_n, Tw) + \alpha_n \|ty_n - w\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|y_n - w\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - w\| - \|x_n - w\| &\leq -\alpha_n \|x_n - w\| + \alpha_n \|y_n - w\| \\
\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} &\leq \|y_n - w\| - \|x_n - w\|.
\end{aligned}$$

Because $0 < a \leq \alpha_n \leq b < 1$, we have

$$\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\| \leq \|y_n - w\|.$$

So, we obtain

$$\liminf_{n\to\infty} \left\{ \left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\| \right\} \leq \liminf_{n\to\infty} \|y_n - w\|.$$

Then we get

$$c \leq \liminf_{n\to\infty} \|y_n - w\|.$$

Since, from (2), $\limsup_{n\to\infty} \|y_n - w\| \le c$, we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(z_n - w)\| \end{aligned} \tag{3}$$

where $0 < a \le \beta_n \le b < 1$.
Hence, we have

$$\limsup_{n\to\infty} \|z_n - w\| \le \limsup_{n\to\infty} \|x_n - w\| = c. \tag{4}$$

By Lemma 2.1, (3) and (4), we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Lemma 3.4 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1). If $0 < a \le \alpha_n, \beta_n \le b < 1$ for some a, $b \in \mathbb{R}$, then $\lim_{n\to\infty} \|tx_n - x_n\| = 0$.*

Proof. Consider

$$\begin{aligned} \|tx_n - x_n\| &\le \|tx_n - ty_n\| + \|ty_n - x_n\| \\ &\le \|x_n - y_n\| + \|ty_n - x_n\| \\ &= \|x_n - (1-\beta_n)x_n - \beta_n z_n\| + \|ty_n - x_n\| \\ &= \beta_n \|x_n - z_n\| + \|ty_n - x_n\| \\ &\le \beta_n \|x_n - z_n\| + \|ty_n - z_n\| + \|x_n - z_n\| \\ &= (1+\beta_n) \|x_n - z_n\| + \|ty_n - z_n\|. \end{aligned}$$

Then, we obtain

$$\lim_{n\to\infty} \|tx_n - x_n\| \le \lim_{n\to\infty} (1+\beta_n) \|x_n - z_n\| + \lim_{n\to\infty} \|ty_n - z_n\|.$$

Hence, by Lemma 3.2 and Lemma 3.3, $\lim_{n\to\infty} \|tx_n - x_n\| = 0$.

We give the sufficient conditions which imply the existence of common fixed points, as follow:

Theorem 3.5 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in Fix(t) \cap Fix(T)$.*

Proof. Assumed that $\lim_{n\to\infty} \|x_{n_i} - y\| = 0$. From Lemma 3.4, we have

$$0 = \lim_{n\to\infty} \|tx_{n_i} - x_{n_i}\| = \lim_{n\to\infty} \|(I - t)(x_{n_i})\| .$$

Since $I - t$ is demiclosed at 0, we have $(I - t)(y) = 0$ and hence $y = ty$, i.e., $y \in Fix(t)$. By Lemma 2.2 and by Lemma 3.3, we have

$$\begin{aligned} dist(y, Ty) &\leq \|y - x_{n_i}\| + dist(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\leq \|y - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| + \|x_{n_i} - y\| \to 0, \end{aligned}$$

as $i \to \infty$. It follows that $y \in Fix(T)$. Therefore $y \in Fix(t) \cap Fix(T)$ as desired.

Hereafter, we arrive at the convergence theorem of the sequence of the modified S-iteration.

Theorem 3.6 *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (1) with $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Since $\{x_n\}$ is contained in E which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in E$, i.e., $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Theorem 3.5, we have $y \in Fix(t) \cap Fix(T)$ and by Lemma 3.1, we have that $\lim_{n\to\infty} \|x_n - y\|$ exists. It must be the case that $\lim_{n\to\infty} \|x_n - y\| = \lim_{i\to\infty} \|x_{n_i} - y\| = 0$. Therefore $\{x_n\}$ converges strongly to a common fixed point y of t and T.

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